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Why Not Be Sensible About Meaning?

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An article in a recent issue of The Mathematics Teacher* admonishes us to be sensible about the meaning we attempt to teach the pupils in our classes in elementary mathematics. It goes on seriously to question both the sense and the utility of teaching meaning, and still further to suggest that it can't be done anyhow.

The main thesis of the article is that performance in mathematics is everything. To this is added the contention that you can't teach meaning and performance together, since to teach one causes you to neglect the other. And providing color and subtle force to the argument throughout is the suggestion that there really isn't any meaning to teach, so why try? The reader is thereby rudely awakened from his easy-chair perusal. He rubs his eyes, and asks himself, Is this an article in The Mathematics Teacher? Can it have been written by a mathematician? I wonder what kind of mathematics *he* teaches? He looks again at the title, and answers, "Sure; why not?"

In trying to be sensible about meaning in mathematics, let us confine our thinking to the arithmetic of the elementary school; for it is at this level that Mr. Buell seems seriously to doubt if any meaning is available, and if any pupil could grasp it if it were. Moreover, Mr. Buell confines his specific criticisms to the recent report of the National Council's Committee on Arithmetic in its emphasis upon the teaching of meaningful arithmetic. It is only at levels above arithmetic that Mr. Buell admits the sense, if not the possibility, of trying to teach meaning at all:

> . . . I would make these (the lower) levels fairly automatic even for able people, and (thus) free their minds for a study of meanings at a higher level, as high a level as students can reach, where more can be done. Transfer of training, if any, is worth much more at the higher level. (pp. 306–307)

Let us, therefore, raise the question whether we can be sensible in erecting the superstructure if we do not think to be sensible in building the foundation.

1. *What is meaning in arithmetic?* Meaning in arithmetic is neither mysticism on the one hand nor an advanced theory of numbers on the other. These are not the horns of the dilemma, as Mr. Buell would have us believe; they belong to some other quite different animal. Moreover, meaning is not a knowledge of application, however "social" or useful. Such knowl-

* Buell, Irwin A., "Let Us Be Sensible About It," The Mathematics Teacher, November, 1944, pp. 306–308.

edge of itself is accepted on faith; it has no certainty except that "it works." It does not provide meaning, though meaning, when given half a chance, leads to it.

Meaning in arithmetic is knowing what one does when he does it. Thus, when a pupil counts a group of two and a group of three together, deliberately with the purpose in mind of counting them as one, he knows what he is doing; and the process and its result have meaning to him. This is quite different from parroting "Two and three are five," and from copying $2+3=5$. The latter set of activities has all the ear marks of addition, but it is not addition to the pupil. To him, it is mere form, or convention, without meaning. It is mere verbalism, pseudo learning of an insidious nature, pseudo because objectively it appears to be the real thing.

Understanding the expression, $2\overline{)10}$, as the question, "how many twos are there in ten?" or as the direction, "divide ten into two equal groups," the pupil has gained meaning for the division required. When he performs the division, as required, whether objectively or as a mere act of memory, he knows what he has done. What is more, he knows what the answer means,—not 5, as in a vacuum, but, in the one case, 5 twos, or, in the other, 5 in each of two groups. This is meaning; not much, to be sure; nevertheless, meaning for what the pupil has as yet learned to do.

"Eight added to six gives 14. That's all there is to it," says Mr. Buell. Unfortunately, in the case of many pupils, Mr. Buell is correct. That is all many pupils are taught. Yet there are pupils who know and this in spite of the teaching they have had, that the answer can be 1 dozen and 2 as well as 14. No, the answer 14 is not all there is to it; had our ancestors not counted their thumbs with the rest of their digits, we should now be writing 16, not 14, as our answer. There can be meaning in the teacher's direction, "add eight and six," provided meaning is put into it. It can mean no more than Mr. Buell suggests, or it can mean "group eight and six into a ten and another group." The question, "Four sixes are how many?" means something: $4\times6=24$; that is all there is to it. No, the answer is 20, if we use twelve as the base of our numeral system; 30, if we use eight as the base. The pupil uses ten as the base; and, depending on the way he is taught, he does so intelligently or otherwise. If he knows what he is doing, his performance has a meaning.

Must the pupil, then, continue to perform in the deliberate round-about way required at the outset? How involved will be his addition if always he must think, "Eight and six are ten and how many?" How involved will be his multiplication if always he must think, "Four sixes are how many tens?" The answer may be had by anyone who pauses to consider that the way by which a thing is learned differs from the way by which it is effectively used. In no adjustment an individual is required to make can his early methods of learning be the same or of the same effect as his later methods of use. In the acquisition of the art of number thinking, there is a pattern of learning, deliberate and groping and round-about; and there is a pattern of use, which may be swift and sure and short-cut. To shape the former pattern according to the requirements of the latter is a distortion of learning. To anticipate the latter in terms of the former constitutes failure to envisage progress in learning and a denial that progress is a possibility. Perhaps no greater service could be rendered teachers of arithmetic than to make clear to them the fact that early methods of learning should be distinguished from later methods of use.

2. *Can pupils grasp meaning?* Whether or not pupils can grasp meanings in arithmetic depends upon where and how they are led to begin. Wait to teach meanings until the later stages of the verbalism of arithmetic are reached; and be too impatient to build step by step from the beginning—the task, then, is extraordinarily difficult, if not impossible. Blame your

failure on the stupidity of your pupils; for you can amass sufficient statistics of dullness and ineptitude to defend your thesis. If your statistics disturb you, look back of them for causes of dullness and ineptitude. You may find that your task is hard, because you have waited too long to begin it. How Mr. Buell "at a higher level" would teach meanings to pupils whose minds are "free" of all underlying meaning and "free" of any previous exercise in building meaning, if he will disclose the plan, would become an interesting lesson in pedagogy.

Whether a pupil can understand a later step in arithmetic depends upon whether he understands the steps preceding. This is so, because arithmetic is more than a collection of useful facts, more than an extended series of things to do. Arithmetic is a science, characterized by its simplicity and consistency. It is a way of thinking about the sizes of groups—about quantities and amounts—rather, a series of progressive, sequential ways of thinking. Omit one of the steps in the sequence, or permit the pupil to by-pass it, and the next in the series is thereby made hard to take. There are no short cuts to meanings in arithmetic. Our question, then, must be, Can beginning pupils in arithmetic grasp meaning at the outset?

The answer is an unqualified affirmative. Counting can mean a way of determining the size of a group; adding can mean counting together; division can mean the separation into equal groups; and so on until, finally, the fraction can mean a relation between one amount and another. The numerals can be learned as means of expressing ideas of amounts, and the places they are written as expressing tens, tens of tens, and so on. Each step of procedure can be learned as a means of finding answers for one's self. Indeed, if started properly, no pupil need be told an answer in all of his arithmetic, either to a combination or to a process. Each new combination or process may be presented as a question for him to answer. Meanings already gained make clear the meaning of the question; and meanings already gained suggest and check his answer.

What has just been said applies not only to bright pupils and ordinary pupils, but also to those who are slow to learn. If there is any difference in the enthusiasm with which pupils respond to meaningful instruction in simple arithmetical meanings, such difference is most noticeable in the case of the slow learners. They frequently give the impression that for the first time in their lives they have discovered in arithmetic something they can do with full confidence within themselves that they are doing it right. It is the writer's observation that wherever elementary-school teachers have made an honest, common-sense effort to build up patiently step by step the simple meanings inherent in arithmetic, all pupils have grasped meaning and profited thereby. Many of his students have assembled evidence that supports the observation. Let those who deny the educability of children assume the burden of proof. Let such analyze out of arithmetic a few of its simple meanings, present them without distracting embellishments carefully and patiently, and locate the children who do not respond.

The question whether pupils in the later stages of arithmetic can grasp meaning has been answered in the case of those pupils who in the earlier stages knew *what* they were doing when they did it. In the case of those later-grade pupils who began as unthinking automatons and parrot-like repeaters of the verbalism of arithmetic, it has been suggested that the acquisition of meanings may be difficult. It may be suggested that the acquisition of meanings by such later-grade pupils is not unduly difficult or an impossibility, depending upon whether meanings appear to them as simple meanings or as complexities.

The meanings in early arithmetic are simple; in later arithmetic, they still are simple, though each in succession is a step beyond those that precede. Why meanings in later arithmetic seem complex and

beyond apprehension is that they are frequently considered in isolation and without due regard for the earlier meanings which give them support. Try to explain division by two-place numbers (long division) to a pupil who up to that point may know how to divide *but does not know what he does when he divides,* and you try the impossible. Try to explain the same to a pupil who both knows *how* and knows *what,* and before you are fairly started he is explaining it to you. To place the former pupil in somewhat the same situation as is the latter, you must delay "long division" until you have taken him back to the beginning of division and taught him the *what* of every preceding step. Try it, and you may be surprised at the speed the impossible becomes the possible. Try it, and you may be surprised at the changed attitude of your pupil toward arithmetic, rather the changed appearance of arithmetic to him. There are teachers who each year take the poorer members of their poorer sections in fifth- and sixth- and seventh-grade arithmetic back for a fresh start at considering what one really does in the simple processes of second- and third-grade arithmetic and then ahead step by step up the trail of meanings. The results are gratifying to all concerned.

3. *What good is meaning in arithmetic?* One may as well ask, what good is arithmetic? since without meaning there is possessed no arithmetic. Or, if it can be conceived that a person possesses an arithmetic without meaning, what does he have and what can he do with it? The answer, however, need not be thus thrown back upon the questioner. It is provided in particular practical form in every classroom wherein the teacher seeks to make meanings clear. The answer is that meaning opens the door to subsequent learning.

a. Meaning provides the key to each succeeding step of procedure. To illustrate, let the pupil understand what he does when he adds tens, and the procedure of subtracting tens becomes thereby considerably simplified. Moreover, as he moves ahead in an intelligent handling of tens, he gains enough impetus to carry him well into, and sometimes through, what otherwise are the mysteries of division by tens (long division). To illustrate further, let the pupil understand what he does when he writes two 1's to represent *eleven,* and the mystery begins to depart from the zero. Let him continue thus to understand, and when he arrives at the study of decimals, he will find little about them that he needs to study.

b. Meaning makes clear the practical application. The essence of the "social" situation to which number is "applied" is the number relation that is involved. Thus, if a pupil considers the situation of nine boys in a co-operative project of buying a baseball, and the meaning of division is already clear, he gains a swift and sure grasp of the situation. What division means makes the social situation of sharing expense significant. Such is the movement of meaning at every stage of practical application. The practicality of the situation is in proportion to the meaning the pupil brings to it. Meaning makes the practical application practical.

Eighteenth Yearbook

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Which Shall It Be: Mechanical Drill or Development in Understanding All the "Whys"?

By Edith L. Mossman

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Shall we try to look squarely at both sides of the issue and do our best to choose the one holding greater power of really good, far-reaching accomplishment? Then must we work with all our might to see it set plainly in all its importance, its fine possibilities before our teacher training institutions.

On the one side there are those who feel that it takes altogether too long a time to present or develop the reasons for the various topics in arithmetic and algebra classes. They believe that it therefore shows more practical, common sense seeing things as they are in this rapidly moving world of ours, to give a sample and have students drill on problems like it until the work can be done quickly and accurately by nearly all the class.

Those taking the opposite side do not deny that presenting or developing the reason in each case does take a lot of time. However they insist that it is worth while to take the necessary time, even at the expense of including fewer topics in the course.

Mechanical drill, following a sample, is given on each topic, over and over again, year after year, not only through elementary grades and junior high, but also in senior high, evening school, college (in private tutoring if not in college classes), in business and trade schools and shops, and recently in many training classes for our armed forces. Here through the years is a vast waste of time, beginning with hardly more than "busy work" that produces pages and pages of this juggling of figures, mechanically following a meaningless sample, calling not at all upon the faculty which study of mathematics is supposed to help in developing, but only upon memory and observation.

A test given to a large number of school children in several states, a number of years ago, brought out two points very plainly: (1) There was an exceedingly high percentage of failures upon every one of the simplest topics taken from the usual course of study in arithmetic. All these had been "*drilled*" upon, many times, term after term. (2) There were very few failures indeed, upon a topic in which the young people would almost surely be interested, would have asked questions, and would have learned quickly and understandingly outside of school.

Why go on, term after term, year after year, with that type of drill under which the "modus operandi" is forgotten over and over again within a few weeks after each periodic spell of "drill" or "review" in unthinking, parrot-like following of a form? No matter how thoroughly something is understood it is likely to be forgotten after a long period of disuse. But if something is never really understood, the forgetting of course takes place after a much shorter period of disuse, sometimes after only a few days or weeks.

Then would it not in the "long run" save time to take the hours necessary in the introduction of each new topic to present or develop it by a method that can result in thorough understanding? Of course, practice is necessary in all arts and skills, but how much more is accomplished by the "meaningful" type instead of in mechanically "flipping the symbols." Then too, these periods of review practice can be spaced farther apart and can be composed of shorter assignments after there has been that first experience of thorough understanding. But each time it is vital to hold the "reason" for each step in the foreground.

Then, too, those in the first camp contend that there are many children not at all interested in knowing the "reasons." It is certainly true that when a teacher of

seventh, eighth or ninth grade tries to develop reasons for processes to which the boys and girls have been exposed a number of times, many of them do appear restless and bored. They have grown accustomed to directions which they can follow quickly and arrive at the result which the teacher or the answer book says is correct.

Sometimes instead of asking the directions for a process such as division of decimals or adding common fractions or finding what per cent one number is of another, the boy or girl helpless before a "thought" or "story" or "situation" problem inquires, "Shall I multiply or divide?" Yes, it is a sad fact, that usually he is perfectly satisfied if the teacher replies in *one word*, and is annoyed if he must take time for discussing the situation in the problem until he sees which to do and why.

But this deplorable state of mind has been brought about in the school room by mistaken methods of teaching. An inteligent boy will misplace a decimal point, giving $2640 as the cost of a ton of coal, or offering $425 as $4\frac{1}{4}\%$ of $100. He pays not the slightest attention to the utter absurdity of these answers. However, on the playground or anywhere *outside of the school room*, if his buddie tells him of a man paying $2640 for a ton of coal, or of one who lent $100 and at the end of a year received back his $100 and as interest for its use an additional $425, there would be a quick retort: "Aw! You're stringin' me!" or "Rubbish! Nobody ever did such a fool thing as pay that much for coal or interest."

At the beginning, primary school children are just as ready with "Why?" in the number work class as they are well known to be everywhere else. That is soon stifled. So much is done mechanically for which they see no reason whatsoever, that gradually they develop an arithmetic school room atmosphere in which they do not try to use their power of reasoning or to give any consideration in common sense. Alas! It is only observation and memory and guessing. Unthinking following of directions. The right answer will pop out at the end. There'll be a good arithmetic mark and promotion to the next grade.

But let me hasten to say that the teachers of arithmetic and algebra are not the ones primarily at fault in this sad state of affairs. They were taught that way themselves in their elementary and high school days. Never in training school were they shown a better method, nor in teachers' meetings were they advised to read professional magazines. Of course there are a few teachers to whom none of this applies, but they are the minority.

Also the programs for arithmetic and algebra classes are so full that the teacher finds difficulty in covering all the topics. Number work should have its start in first grade. The little folks love it, and are ready to do some thinking work on some of it right then. There should be fewer new topics each term, each presented or developed by a method that makes understanding possible. Then there should follow plenty of thinking, meaningful practice to produce a high degree of accuracy and some speed.

There are those in the Progressive Education movement, who ruthlessly put fingers on these sore spots of senseless, mechanical drill. But the pity is that many of them went to another extreme and developed new weak spots of no drill or practice on fundamentals, no consistent building up of this systematic, logical way of thinking in the world of mathematics, and no foundation in basic, unchanging principles.

There is a third point to be presented in this article. Some people, yes some teachers have said or written that it is not valuable to teach the reasons; that those who need the processes will be using them constantly and so will keep them fresh in memory; that those who will not need to use them—well, what's the odds if they do forget?

The National Council of Teachers of Mathematics held its twenty-third annual meeting in San Francisco in February, 1942. Among the speakers of the second day was J. Kadushin of the Education

Department Industrial Relations Office, Lockheed Aircraft Corporation, Burbank. His topic was: "Mathematics in Present Day Industry." Near the beginning of his address, he made this point: "There are many who think that in industry are found a large number of mundane jobs that require no higher science of mathematics, that use only "rule of thumb" in a continuous same operation, just blind application. But when comes the slightest deviation, as often does occur, the workman is lost! He needs understanding of mathematics, and has it not!"

Later Mr. Kadushin spoke of this: "Some can handle similar jobs but need to develop new processes, new methods. They cannot deviate from the beaten path because of *weakness in mathematics*, in fractions, etc. Among the unskilled, thousands of hours are lost because of these workmen's lack of knowledge of common and decimal fractions—thousands of dollars are lost through inability to change quickly and accurately from decimal to common fraction form and vice versa. Among the semi-skilled on the assembly line and installations where there are more and more complicated situations, there is ever increasing need of mathematics. Without this knowledge, great delay results." The speaker declared: "Pupils are *exposed* to mathematics in the schools but are not *taught* it."

The experience of a mathematics coach, at any college or high school either, shows what a tragedy is the lack of *understanding* of simplest processes. For a number of years, I did a lot of coaching, taking those who needed help from college young people and from both senior and junior high school, also from elementary grades. To those older students, I taught very little indeed, of college mathematics topics. Instead, they needed the foundation of understanding in common and decimal fractions, in percentage and mensuration from arithmetic; in the formula and other kinds of equations, in exponents and several more topics from algebra; in basic points of plane geometry; and in simplest elements from trigonometry. After they had these clearly in mind, they could go on successfully in their college mathematics without having to pay a coach for help.

In all this kind of work with those of widely varying ages, I kept a list of the type of mistakes made. It was astonishing to me then to see how often all along the line from the youngest through the oldest students, these mistakes could be classified under disobeying one or more of the comparatively few fundamental, very simple principles.

For instance:

4 of *them*+5 of *them*=9 of *them*
The *very same* "them" in all 3 places.

9 of *them*−5 of *them*=4 of *them*
The *very same* "them" in all 3 places.

Every time an intelligence test is given at college, a surprisingly large number in answering the arithmetic questions give this sort of thing

$$\frac{2}{3}+\frac{3}{4}=\frac{5}{7}\,.$$

One who has been held throughout the grades and high school to see clearly and apply, wherever possible, the principle stated above, to be sure that he never violates it in any circumstance of addition or subtraction, would say, "You can't add pollywogs and hair ribbons. Adding 3rds and 4ths and getting 7ths is absurd. It can't be done. 'Tis like saying, '2 apples +3 chickens = 5 dominoes'."

Of course, others having been allowed to use the word "cancel" with no thought of correct meaning they will violate another simple principle by crossing out anywhere, symbols that look alike. Therefore

$$\frac{2}{\cancel{3}}+\frac{\cancel{3}}{4}=\frac{2}{4} \quad \text{or} \quad \frac{\cancel{2}}{\cancel{3}}+\frac{\cancel{3}}{\underset{2}{\cancel{4}}}=\frac{1}{2} \quad \text{or just 2.}$$

In like manner:

$$\frac{\cancel{a}+b}{\cancel{a}+c}=\frac{b}{c}\,.$$

These errors appear because forms or directions have been followed mechanically without understanding of fundamental principles involved. There are *many* points all through arithmetic and first year algebra where that first principle applies. There is not time here for other illustrations.

The few simple principles thoroughly understood from the beginning of number work in elementary grades would prevent most of the vast number of usual mistakes made every term in the classes of arithmetic, algebra and higher mathematics.

One year when teaching in an evening school, the principal asked me to take a class of men in "Business Arithmetic." Of course, every wide awake teacher recognizes the importance of practical applications. I had always been careful to talk with men in offices and factories about their methods in calculations. In our text books are sometimes found forms that are laughed at by those out in the world of practical affairs.

But I had not had any business experience. Also I knew that the place in which I could do the best job possible was in making simplest fundamentals understood, so that with that foundation, boys and girls could go out and successfully apply those principles whether in college courses or in offices or stores or factories.

So I was asked if I were willing to go in and interest those men for one evening, or until an experienced teacher of "Business Methods" could be found. The room was full of intelligent, successful looking men. I was perfectly frank with them about the situation. But one said, "Maybe you can give what I want. I want to understand about finding what per cent one number is of another. In my school days, we divided. But how do you *know* which to divide into the other? I just try whichever way it happens. If the result is absurd,then I try it the other way. But I want to *know* which to use for the divisor and *why.*"

Right away, a second said, "My business is totally different from his, but I want to know the same thing." Then up spoke a third: "Those two are much better off than I. If they divide and get an absurd answer they then exchange the numbers. But I am not sure where to put the decimal point in the answer, and so do not know whether or not the answer is absurd."

Well, those men asked that I help them the entire term. They made out a list of topics. Each evening we worked first for the *understanding* of the simple principles involved. After that we made practical application in all sorts of problems that they gave for us.

I did not present the finding of rate per cent by method given in our text books. They agreed with me in thinking the common fraction method much better. However we did discuss the usual division method of our texts so that they came to see which number to use as divisor, were that method being employed.

And what about those who will not use the processes and so may as well be allowed to forget them? In the first place men and women in later life often have needs quite different from what they expected. And in the second place, why not teach by such a method that the *experience* of learning would have inherent in it definite value for every one in the class regardless of what later life might bring forth?

The old statement about mental discipline was extreme and fanciful and was "exploded" some time ago. Young people can move through their courses of arithmetic, algebra, geometry, etc., making *A*'s and *B*'s, but rarely using any other power than those of memory and observation. Then of course, there is no development of logical reasoning power.

But psychologists agree that habits and attitudes are transferable, especially so if the student is kept conscious of the possibility and desirability of such transfer. If given opportunity to do so, young people in both junior and senior high school classes will produce surprisingly fine lists of "Desirable Habits and Attitudes to Be

Formed or Strengthened in a Mathematics Course."

A boy who had had a great struggle through general mathematics and first year algebra enrolled in a plane geometry class, but early in the term persuaded his parents to let him drop that subject and give up the idea of going to college. Shortly afterward, he was employed for after school hours in work that required almost no mathematics at all. He progressed very rapidly, received promotions after surprisingly short periods. When drafted for our armed service, he was chosen for special, expert work that gave great joy to him and to his parents. In talking about these things, he said that the two years spent in general mathematics and algebra far from being wasted prepared him for the possibility of those promotions and the fine position in the service of his country. He expressed appreciation of the teacher who in those years showed him how to develop habits and attitudes that were necessary for that succession of events.

Countless numbers of illustrations of this type from various walks of life could be given if time permitted. An employer seeking a young man will, other important points being equal, choose the one with good training in mathematics, even though the position to be filled will require no solving of problems in arithmetic, algebra or etc.

Then shall we ask: Is it not true that much more is accomplished in teaching mathematics if the method is used which makes possible thorough understanding of each step, full possession of fundamental principles and conscious development of desirable habits and attitudes? Is it not true that then (1) Time is saved rather than wasted in the "long run?" (2) The unfortunate state of mind of not caring for reasons is corrected? (3) The experience of *really learning* mathematics is valuable in all walks of life?

And here is one more phase of value to be added to that third point: Both young people and adults who have had great difficulty in mathematics classes, always receiving low grades, often develop an inferiority complex. They think they must not be as *smart* as those who receive better marks. When fundamental principles are made plain to them and they find they, too, are able to attain a certain degree of success in these courses, they have a greatly improved outlook on life. They know how to take hold of various things in a more systematic, advantageous manner; they can do more different things well themselves, besides finding more freedom from embarrassment, more real pleasure in following the accomplishments of others

The Commutative Law

By Elmer B. Mode
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The commutative laws of algebra are familiar to every well trained teacher of mathematics. Briefly, they state that

$$A+B=B+A,$$

and that $$A\times B=B\times A.$$

These are the commutative laws of addition and multiplication respectively. Obviously $A-B\neq B-A$ and

$$\frac{A}{B}\neq\frac{B}{A},$$

in general, so that the commutative law is not valid for the inverse operations of subtraction and division.

Ordinarily little work is undertaken in high school courses in connection with these and associated laws. The practical aspects of mathematics too often crowd the cultural out of the class room. However, it is heartening to observe renewed attempts to acquaint the more advanced high school student with a few of the more fundamental characteristics of a mathematical science. The structure of mathematics can be taught well, not only by means of elementary geometry with its undefined terms, assumptions, definitions, proofs, and theorems, all explicitly stated, but often better by means of algebra, where the abstract nature of the subject enables one to apprehend the formal structure more readily.[1] Nevertheless, the postulates of elementary algebra are still very much unknown, even to the above-average senior high school graduate. These postulates offer much of genuine interest, and a skilful teacher needs mention but a few occasionally in order to arouse a warm interest in them. The commutative law has connection with many non-mathematical relationships in life, and it has practical bearings on algebraic technique that are not always recognized.

First let us examine the phrase "commutative law." It is a common one in American text books on college algebra. One may properly ask, "Is this a *law* in the same sense as that in which we speak of the *law* of exponents, $a^m a^n = a^{m+n}$?" The word *law*, as used in mathematics, sometimes has a detrimental effect on our teaching. The law, $a^m a^n = a^{m+n}$, for positive integral exponents, is a *theorem*, easily proved. The law $a+b=b+a$ for *real quantities* is a postulate, that which is assumed. As a matter of fact, the commutative law in elementary mathematics is sometimes a theorem and sometimes a postulate. Thus the commutative law for *positive integers* is a theorem capable of a simple demonstration when a positive integer is defined in terms of the primitive notion of a class.[2] Similarly the commutative law of addition (or multiplication) for complex numbers, that is, numbers of the form $a+bi$, is a theorem whose proof is based upon the commutative postulate for real numbers. Would it not be a step forward in our teaching to distinguish, where feasible, between postulate and theorem, by speaking of the commutative postulate or the commutative theorem?

It is time now that we state the commutative law in a more general form.

If A and B are two objects and $\bigcirc$ is an operation, then

$$A\bigcirc B=B\bigcirc A$$

Thus if $A=a$, a real number, and $B=b$, a real number, and if $\bigcirc$ means $\times$, the usual symbol for multiplication, then

$$a\times b=b\times a$$

[1] See E. V. Huntington, *The Fundamental Propositions of Algebra*, No. IV in Monographs on Modern Mathematics, Edited by J. W. A. Young, Longmans, Green, 1927, p. 152.

[2] See John W. Young, *Lectures on Fundamental Concepts of Algebra and Geometry*, Macmillan, 1920, p. 100.

is the commutative law (postulate) for the multiplication of real numbers.

If $A=a+bi$, a complex number, and $B=c+di$, a complex number, and $\bigcirc$ means $+$, the usual symbol for addition, then

$$(a+bi)+(c+di)=(c+di)+(a+bi)$$

is the commutative law (theorem) for the addition of complex numbers.

It has long been recognized that the notion of *operation* can be replaced by a more general one, that of *relation*. The operation of combining two objects in order to obtain a third object merely establishes the fact that the objects combined bear a certain relation to the object produced by the combination. In more abstract terms, to assume

that $\quad a\bigcirc b=c$

and that $\quad b\bigcirc a=c,$

so that $\quad a\bigcirc b=b\bigcirc a,$

is merely to state that the relation, R, existing among the ordered objects a, b, c is such that

$$R(a, b, c)=R(b, a, c).^{3}$$

Inasmuch as the third object, c, does not explicitly appear in the common form, $a\bigcirc b=b\bigcirc a$, of the commutative law, we shall write the equivalent equation

$$R(a, b)=R(b, a).$$

Let us then state, once more, in words, a more general form of the commutative law.

If an object A is related to an object B in a certain way, then the object B is related to the object A in the same way. In other words, the commutative law affirms the invariance of the relationship under an exchange of the objects. It may also be regarded as establishing a type of symmetry. Consider a few illustrations.

(1) The chemical combination of two elements is commutative. Thus, hydrogen combined with chlorine is the same as chlorine combined with hydrogen $H+Cl=Cl+H$.

(2) Parallelism is commutative. If line a is parallel to line b, line b is parallel to line a.

(3) The spouse relationship is commutative. If A is the spouse of B, B is the spouse of A.

If is well also to consider a few non-commutative relations.

(4) The division relationship is not commutative.

$$\frac{A}{B}\neq\frac{B}{A} \quad \text{in general.}$$

(5) The sister relationship is not necessarily commutative. If A is the sister of B, B is not necessarily the sister of A, for B may be A's brother.

At this point we should recall that the *objects* referred to do not have to be physical objects such as apples or men, nor even abstract numbers, such as A's and B's. They may themselves be operations, such as rotations of angle, displacements of points, finding of logarithms, and so on. Thus the rotation of a line segment AB about A through an angle α and then through an angle β yields the same as a rotation first through angle β and then through α.

The commutative law is rarely mentioned in high school work below the advanced (college) algebra level, yet it seems to be a proposition so fundamental and self-evident that its too universal acceptance is a pitfall for many a pupil. Yet you may ask, "How can it be a pitfall when it is rarely taught or even mentioned?" The answer is simple. It is intuitively, if vaguely, accepted by many students and unconsciously extended to many operations. How else can you explain such misstatements as the following?

(6) The square of the sum of two numbers equals the sum of their squares,

$$(a+b)^2=a^2+b^2.$$

[3] See M. Bôcher, "The Fundamental Conceptions and Methods of Mathematics," *Bulletin of the American Mathematical Society*, Vol. 11, p. 129.

(7) The logarithm of the reciprocal of a number equals the reciprocal of the logarithm.

$$\log \frac{1}{N} = \frac{1}{\log N}.$$

(8) The sine of the sum of two angles equals the sum of their sines,[4]

$$\sin (A+B) = \sin A + \sin B.$$

We need not heap blame entirely on the head of the pupil for misusing the commutative property. Similar blunders have been made in the past by eminent mathematicians. Thus some have assumed, erroneously, that the limit of a function of two variables, x and y, would be the same whether x approached its limit first and then y, or whether y approached its limit first and then x. That is, they assumed that

$$\lim_{y\to b}\left[\lim_{x\to a} f(x, y)\right] = \lim_{x\to a}\left[\lim_{y\to b} f(x, y)\right].$$

This is not always true as the following classic example will suffice to show.

$$\lim_{y\to 0}\left(\lim_{x\to 0}\frac{x-y}{x+y}\right) = \lim_{y\to 0}\left(\frac{-y}{y}\right)$$
$$= \lim_{y\to 0}(-1) = -1$$

but

$$\lim_{x\to 0}\left(\lim_{y\to 0}\frac{x-y}{x+y}\right) = \lim_{x\to 0}\left(\frac{x}{x}\right)$$
$$= \lim_{x\to 0} 1 = 1.$$

Thus we obtain different limits -1 and $+1$, depending upon the order of the operations of passing to the limit.

The untruths (6), (7) and (8) above are obtained by the mere interchange of two words, exactly the germ of the commutative law. If we take (6) for example, and let A represent the operation of addition, S the operation of squaring, then (6) states erroneously that $SA = AS$. Redefine S as the operation of finding the sine and you have the same misstatement for (8). In (7) if R is the operation of finding a reciprocal and L the operation of finding a logarithm then $RL = LR$ is not true. Such typical recurrent mistakes may often be forestalled by a judicious discussion of the commutative law and its misapplication. At an appropriate time the commutative postulate of addition and multiplication for elementary algebra may be explained briefly. It is important to give examples of commutative laws which do not hold, such as those in (4), (5), (6), (7), (8), and in the list given below. A few simple exercises, particularly those of a non-mathematical nature, create interest and convey meanings which often the mathematical ones fail to do. The writer has tried out these devices in his own college freshman classes and, when opportunity offered itself, in classes of (mathematically) low grade high school seniors. Even the dullest student saw the point involved at once and in some cases certain typical errors were eliminated once and for all. Here are some simple but useful exercises.

Does the commutative law hold for the following?

(9) Subtraction. Does $a-b=b-a$?

(10) Raising to a power. Does $a^b=b^a$?

(11) Marriage. If John marries Susie does Susie marry John?

(12) Cousins. If Tom is the cousin of Bill, is Bill the cousin of Tom?

(13) Is the son of his father the father of his son?

(14) Is the line drawn from A to B the same as the line drawn from B to A?

(15) If one spherical triangle is the polar of a second, is the second the polar of the first?

(16) Is the limit of a quotient equal to the quotient of the limits?

(17) Is $\sin \frac{1}{2}x = \frac{1}{2}\sin x$?

(18) In a football game when it is first down, is a three-yard gain through the line followed by a successful 20-yard forward pass equivalent to a 20-yard forward pass followed by a three-yard gain through the line?

[4] This may also be explained as a misuse of the distributive law $a(b+c)=ab+ac$, where "sin" is erroneously treated as a number.

(19) Is the gravitational attraction of the earth for the moon equal to the attraction of the moon for the earth?

(20) Is the derivative of the sum of two functions equal to the sum of the derivatives of the two functions?

(21) Is the derivative of the product of two functions equal to the product of their derivatives?

(22) What is meant by a *commutation ticket* on the railroad?

(23) Is the binomial expansion commutative, i.e., does $(a+b)^n=(b+a)^n$ for positive integral n? For other values of n?

(24) "Congratulations, professor, I hear that your wife has presented you with twins. Boys or girls?"

"I believe one is a boy and one is a girl—but it may be the other way round."

Query. What law has the professor forgotten?

A Guiding Philosophy for Teaching Demonstrative Geometry[1]

By Morris Hertzig

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In his recent book, *Persons and Places*, Santayana makes the following statement about the instruction he received in mathematics:

"If my teachers had begun by telling me that mathematics was pure play with presuppositions, and wholly in the air, I might have become a good mathematician, because I am happy enough in the realm of essence."

Despite Santayana's criticism, this is precisely the concept of mathematics that we in the high school must deliberately avoid. Our pupils constantly demand that we make the subject real and practical. They insist upon knowing its uses and importance. To be sure they are not all Santayanas, but even Santayana's conception of the nature of mathematics is open to criticism. Russell, for example, attempts to define numbers so as to make their connection with the actual world of countable objects intelligible. He feels that our whole concern with the axioms of arithmetic has to do with the application of numbers to empirical material. Students should be given the same feeling about the axioms of geometry which provide the implicit definition of all undefined terms such as "line," "point," etc. The axioms are set up so as to make them agree fairly well with the properties of physical objects such as a pencil dot or a taut thread. Our major concern with the axioms and concepts of geometry has to do with their applications to empirical material.

But even if Santayana is correct in his conception of mathematics as "pure play with presuppositions and wholly in the air," it is an almost impossible task to teach this concept of an abstract mathematical science on the high school level. It becomes particularly confusing to do this when at other times we present mathematics as the handmaiden of the sciences or even the queen of the sciences.

One of the earliest opportunities we have to give pupils an insight into the nature of mathematics as a postulational system occurs in the course in demonstrative geometry given to pupils who are 14 or 15 years old. It is here that we commit several glaring errors that arise from our unwillingness to adopt a definite philosophy of the subject to be taught.

In our introduction to demonstrative geometry, points, lines, triangles, etc., are considered as real objects. The necessity for establishing absolute truths concerning these real objects is the common motivation for undertaking the demonstration of propositions that can otherwise be obtained experimentally. We lead our pupils to believe that observation always carries with it an element of doubt.

This attitude gives a false conception of the power of a demonstration, the significance of postulates, and the intended goals of experimental procedures.

We forget that without assumptions there can be no proof.

We forget that the discovery of non-Euclidean geometry destroyed the conception of the axioms of Euclid as the immutable mathematical framework into which must be fitted our knowledge of space, that reasoning from them does not necessarily lead us to absolute truths concerning the nature of space.

We forget that experimental procedures do not seek to establish absolute truths, but merely enable us, in the language of Dewey, to make warranted assertions. "The 'settlement' of a particular situation by a particular inquiry is no guarantee that that settled condition will always re-

[1] Read at the third annual meeting of the Metropolitan New York Section of the Mathematical Association of America, at New York University on April 22, 1944.

main settled. The attainment of settled beliefs is a progressive matter; there is no belief so settled as not to be exposed to further inquiry. It is the convergent and cumulative effect of continued inquiry that defines knowledge in its general meaning. In scientific inquiry, the criterion of what is taken to be settled, or to be knowledge, is being so settled that it is available as a resource in further inquiry; not being settled in such a way as not to be subject to revision in further inquiry."[2]

Later in the course, we feel that we must introduce the concept of mathematics as "pure play with presuppositions and wholly in the air," forgetting entirely our earlier appeal based on the necessity for establishing truths about physical points and lines. Points and lines then become fictions. The postulates are arbitrary rules. The fascination lies in mere manipulation of our arbitrary rules concerning fictions. It is an interesting game we play, but is it any wonder that these immature minds of 14 or 15 do not have the slightest conception of what we are trying to do?

There is a beginning, in this city to do away with this confusion. This new program is based on our unwillingness to attempt the impossible any longer. It is based on a very humble conception of geometry as an empirical science of space. The demonstrative aspects of the course are included to initiate pupils into the techniques of scientific method and not into the free and rare atmosphere of pure mathematics.

The course of study is so arranged today that many of the propositions of geometry are obtained experimentally during the 7th, 8th and 9th years. On this level, we are making the work experimental rather than intuitive. The logic of experimental inquiries must therefore be observed. What we have heretofore called intuition in geometry is only the beginning of experimental inquiries. The scientist is never satisfied with a mere guess. The guess must be submitted to the test of observation. For this purpose, the scientist has devised tools of ever increasing precision. For the student of geometry in the 7th, 8th, and 9th years, the tools of observation are rulers, protractors, squared paper etc. Properties of figures are not usually discovered by means of these instruments. Properties suggest themselves to the pupils. The instruments are used to test these suggestions. The test must be deliberately and carefully planned since it constitutes an important phase of scientific method.

Experimental inquiries follow a very strict logic of their own. Our colleagues, the teachers of science, try to impress this on our pupils. Yet how often do we in the mathematics class fail to adhere to the pattern of sound experimental procedure. Most of us do teach the necessity for an adequate number of representative samples in order to arrive at a generalization. But there are other factors to consider. Alternative possibilities sometimes have to be excluded. The question must frequently be investigated on the negative side. The aspect of control is also important. For example, in testing whether or not the base angles of an isosceles triangle are equal, when preparing our materials, we control the length of two sides and then note the relationship between the angles over which we exercised no control. A controlled experiment not only seeks to determine the possible "cause"; it also seeks to isolate it. And so in the experiment just mentioned we can also control the area and the vertex angle to show that these do not affect the conclusion. The control, therefore, seeks out those things that need not be controlled.

During the tenth year, there is ever present the problem of motivating the demonstration of propositions which students are willing to accept on the basis of observations they have made. Why should these propositions be reconsidered from the demonstrative point of view?

Pupils must be led to understand that scientific method does not end with the generalization obtained by induction. Merely to observe a fact is not to have scientific knowledge. The scientist always

[2] Dewey, *Logic, The Theory of Inquiry*, p. 8.

seeks to account for the facts observed—to explain them. His great objective is to explain them in terms of a very few simple principles.

When Archimedes formulated his law concerning a solid immersed in a fluid he was not satisfied to note what appeared to be a brute fact. His major concern was to account for the phenomenon, to explain why it occurred. This explanation consists in exhibiting necessary relations between the nature or definitions of fluids and the nature or behavior of solids immersed in them. And so Archimedes turned his attention to the investigation of fluids to discover a generalized conception of them that would account for the observed behavior of solids immersed in them. As a result of this investigation, he formulated a treatise on fluids which begins with a postulate that serves to define the nature of fluids. From this, he was able to demonstrate the theorem mentioned.

"In arriving at a scientific law there are three main stages: the first consists in observing the significant facts; the second in arriving at a hypothesis, which, if it is true, would account for these facts; the third in deducing from this hypothesis consequences which can be tested by observation."[3]

This method is revealed on a grander scale in mechanics. Here Newton was able to arrange the law of falling bodies, the laws of the tides, the law of the pendulum, Kepler's laws, into one systematic organization exhibiting necessary interconnections and all stemming from his law of gravitation and laws of motion.

"Science, in its ultimate ideal, consists of a set of propositions arranged in a hierarchy, the lowest level of the hierarchy being concerned with particular facts, and the highest with some general law, governing everything in the universe. The various levels in the hierarchy have a twofold logical connection, traveling one up, one down; the upward connection proceeds by induction, the downward by deduction. That is to say, in a perfected science, we should proceed as follows: the particular facts, *A*, *B*, *C*, *D*, etc., suggest as probable a certain general law of which, if it is true, they are instances. Another set of facts suggests another law, and so on. All of these general laws suggest, by induction, a law of higher order of generality of which, if it is true, they are instances. There will be many stages in passing from the particular facts observed to the most general law as yet ascertained. From this general law we proceed in turn deductively, until we arrive at the particular facts from which our previous induction had started."[4]

In a geometry course considered as an empirical science of space we can develop an appreciation of this ideal in science and permit our pupils to build several segments of the grand structure of geometry. The objective of the 10th year should be to combine all the apparently isolated facts and generalizations discovered empirically in the 7th, 8th, and 9th years into one integrated system. From all the apparently unrelated generalizations is it possible to abstract certain generalizations which will account for all the observed facts? These are the postulates of a geometry of Euclidean space. And there must be a good many of these postulates on the 10th year level. But these postulates are not arbitrarily selected and are not a priori. For our students they must be the results of experiences concerning space. The postulates are not arbitrarily selected because their selection and retention depend on how successfully they can be used to account for all the facts that set the whole inquiry into motion. They are inductively obtained in the course of inquiry into the nature of space and are retained because they are found to account satisfactorily for the observed facts.

Viewing geometry as an empirical science is very repugnant to some teachers of mathematics. For them, geometry must

[3] Bertrand Russell, *The Scientific Outlook*, p. 57.

[4] *Ibid.*, p. 58.

remain pure. While mathematics might be able to attain complete freedom from all existential reference of even the most indirect, delayed, or ulterior kind, it is difficult to see how geometry on the high school level can be presented as having such freedom when it is aimed at existential applications, and its contents, while abstractions, have their source and end in concrete situations. The mathematician is free to consider a geometry as defined by any set of consistent axioms about meaningless entities, but his investigations are useful to the scientist and meaningful to students only if these axioms correspond to the physical behavior of real objects in the real world. Pure mathematics, when free of the necessity of existential reference, gives wide opportunities for existential applicability. The range of existential applicability increases with its abstractness. But students of 10th year mathematics are too immature to grasp the notion of an abstract mathematical science. We must be content with giving them a firm knowledge of a postulational scientific system. This is the essence of the mathematical method. The course in geometry offers the best opportunity on the high school level to study the postulational method, and one of the few opportunities to study the complete scientific method.

In his course in geometry, the student has an opportunity to engage in experimental inquiries and study the logic of sound experimental procedures. He also learns that an induction obtained as a generalization from several observations is insufficient to give scientific knowledge of the principle observed. The facts must be explained in terms of simple, general principles. When this is done, what previously appear as a number of isolated and independent phenomena are revealed as parts of an interconnected system.

The deductive aspect of the scientific method also develops an appreciation of the further advantages to be secured in seeking the implications of a set of facts. In this way, we can move from what is known to what is not known. Deduction, plus later confirmation by experiment, is an important device for extending the limits of knowledge. Students are impressed by stories of the discovery of the planet Neptune and Maxwell's deductions concerning the existence of electromagnetic waves. They are also thrilled by their own deductive discoveries concerning the nature of space.

If the philosophy outlined here is correct, then the following charge by E. T. Bell is answered:

"Any one who was subjected to elementary geometry when his infantile brain was as unripe as a green walnut will recall the protracted misery he endured. Through stupid exercises of cutting out cardboard squares, rectangles, and circles, and measuring and weighing them, he struggled to placate his teacher by 'rediscovering' the idiotically simple 'rules' for finding the areas of such things. As scissor-and-balance gymnastics these tortures may have been excellent initiation to the mysteries of a school laboratory in physics. As an introduction to mathematics, and in particular to geometry, they were silly, incompetent, immaterial, and irrelevant."[5]

Bell is not entirely correct. Such activities can be very wise, competent, material, and relevant as an introduction to mathematics and in particular to geometry. Geometry can best be understood by high school students as an empirical science. One of its greatest contributions is to explain, in a very simple manner, the nature of scientific method. And in the ordered relations that the propositions of a unified science sustain to one another, lies the very essence of the postulational nature of mathematics.

[5] E. T. Bell, *Handmaiden of the Sciences*, pp. 6, 7.

A Program for Individualizing Instruction in Mathematics

By Daniel W. Snader
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There is nothing so unequal as the equal treatment of unequals

The Present Scene

One of the great problems facing our schools today, and our nation of tomorrow, is to re-vamp our instructional techniques and administrative procedures to provide educational opportunities for our youth which are commensurate with their abilities, interests, and needs.

In a Democracy we believe in an educated proletariat. Equal rights and opportunities should be afforded the children of all the people of our land. We American educators like to think that we have been providing these opportunities. But have we? From the days of Comenius, LaSalle, Lancaster, Harris, Search, Kennedy, and others to the present era of the Morrison plan, the Dalton school, and theWinnetka system, we might trace the ebb and flow of the educational tide in its search for more enlightened procedures in adapting instruction to the individual child. Yet today it is generally admitted that our most neglected pupils are those brilliant boys and girls who possess the qualities of leadership which a nation can ill afford to waste or ignore. Prior to the war there was a dangerous tendency to level-downward the caliber of class work to meet the tempo of the average child, while the superior pupils were unchallenged and unproductive. The recent reports from our Army and Navy officials give us further evidence of this deplorable condition.

To point out the weaknesses of our present system of mass education will not in itself bring about the much needed reform. Merely to recognize that pupils differ widely as individuals does not meet the challenge. Wise and thoughtful class-room experimentation is our only hope.

How can teachers of mathematics meet this problem of adjusting instructional materials and teaching techniques to the abilities, needs, and interests of the individual pupils without creating insurmountable administrative difficulties?

Investigations show that school organizations, courses of study, and methods of teaching, though giving "lip-service" to the contrary, have been developed and operated in such a way as to imply that all pupils may be treated alike. The exceptionally bright pupils have rarely been producing to the limits of their capacities, and many of the "dull-normals" have been literally "forced" along by *standard rates of progress through standard courses of study.* How can the schools justify this enormous waste of time and energy? How can this situation be improved?

The Milne School Plan for Individualizing Instruction

This review is based upon the report of a project entitled: "A Program for Individualizing Instruction in Senior High School Mathematics," accepted by the Advanced School of Education, Teachers College, Columbia University, in partial fulfillment of the Ed.D. degree.

During the past fifteen years the writer has studied and experimented with classroom procedures, materials of instruction, pupil difficulties in learning, test construction and design, and in providing for administrative services and modifications required to promote a program of individualized instruction in secondary school mathematics. The experimentation was carried out by the writer in the states of Pennsylvania, Ohio, and New York under a variety of different conditions, among thousands of pupils, and tested by both

local and state examinations. The program can be seen in operation in the Milne School of New York State College for Teachers, Albany, N. Y.*

A Statement of Objectives

Among the principal objectives of individualized instruction in the program of mathematics are:

1. To extend individualization upward into the high school level of instruction.

2. To develop a procedure for effectively meeting such standards as set by New York Regents and other state and national committees—on an individualized instruction basis.

3. To provide for individual progress of pupils according to their interests, needs, abilities, and initiative.

4. To provide the essential definiteness which facilitates effective learning activity.

5. To develop a keener sense of individual responsibility, industry, and accomplishment.

6. To cultivate good mathematical habits, such as neatness, accuracy, relational thinking, correct reasoning, generalization, etc.

7. To cultivate the ability to apply mathematics to problems of business, science, industry and life situations.

8. To prepare students for futher study in advanced mathematics.

9. To economize in effort and time, both on the part of pupils and teachers.

10. To strengthen and enrich the curriculum for the potential leaders of the future.

11. To reduce the number of "failures" to a minimum.

12. To make *guidance of pupil activity a professional virtue* and *teacher-telling a professional vice.*

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The Minimum Requirements For Making the Program Effective

In preparing the American youth for the role it must play in the world of tomorrow, we need to re-think and harmonize the purposes of the administrative and instructional functions of our schools.

I. *Individualization of instruction is administratively* possible.

Among the views often expressed by administrators we find: Those who believe that the need justifies a radical re-organization of educational procedures, leading, in consequence, to the virtual abolition of class teaching. To this the writer does not subscribe.

2. Those who believe that class instruction should be retained but that adaptations should be made by which the inequalities among pupils can be dealt with on a semi-individual basis. Ability grouping, elastic assignments, flexible standards, frequent promotions or a combination of these are among such proposals.

II. *Individualization is instructionally possible.*

An individualized procedure of instruction, to operate efficiently in the ordinary class-room, must be predicated upon the theory that:

1. *Learning is an active process.*

2. *Instructional materials must be written to the learner.*

3. *The materials of instruction must be written in the language which is understood by the learner.*

4. *The usual class-room controls must be adjusted so as to be conducive to individual progress at varying rates and levels of achievement.*

III. *In making individualization of instruction possible we need to modify and adjust the schools' curricular offerings to the needs, interests, and capacities of the individual pupils.*

IV. *Teaching techniques should be based upon recent findings in the field of the psycholoy of learning.*

V. *We must create new instruments of instruction which are consistent with the philosophy and psychology of individualization. Among the special instruments of instruction required for this system of individualization are:*

1. *The explanatory materials* containing the inter-active teacher-pupil development of concepts and meanings, a large variety of fully solved illustrative examples, up-to-date problems, and practice exercises. This calls for a modification in the philosophy of textbook writing. Explanations must be written to the pupils in clear simple language, approaching as near as possible the type of explanation which a teacher might use in regular group instruction.

2. *Tests of a new design,* both in style and purpose, for facilitating this important phase of the teaching-learning process.

3. *A marking scheme* which permits some *flexibility* within a framework of *acceptable individual standards.*

4. *A sequence of mathematics courses* to provide an opportunity for the study of mathematics as a continuous function throughout a given period of time.

How the System Operates

A detailed description of the class procedure is beyond the scope of this brief report. In general, this is how the plan works in Intermediate Algebra.

1. The teacher explains to the pupils the basic purposes of the course and the techniques by which they can be realized.

2. Before they start with Unit I, they must pass the "Inventory Tests" covering a review of Elementary Algebra. Thus, the foundation is checked before building the "super-structure."

3. During the period of the "inventory," the procedure is semi-individual. The teacher does the necessary re-teaching individually, or in small groups as the occasion requires.

4. As soon as a pupil has successfully completed these reviews he is permitted to begin his study of Intermediate Algebra. The teacher now gives him special guidance and instruction on:

a. How to study the Unit Explanations.

b. How to use the practice materials and problems to develop facility, skill and power in problem solving.

c. How to use the Unit Tests for preliminary personal check-ups on his understanding of the successive units.

d. How to become eligible for the Unit Test (under supervision of the teacher) from which achievement levels are determined.

e. How to interpret the test results, correct errors, and record test marks on individual achievement record forms, etc., etc.

f. The teacher now assumes the role of a consultant, guide and counselor. He moves freely about the room giving aid here, encouragement there and always with the thought of teaching pupils to help themselves, become independent thinkers and assume their proportionate share of the responsibility for their education. This free and friendly interaction of the pupil-teacher-group is the best possible relationship for effective learning.

g. Pupils take Unit Tests when *they* are ready for them. They are reminded from time to time of the tentative time schedules for finishing a course in one semester or one year so that they may budget their time accordingly.

h. If the level of achievement is inadequate to satisfy their own desires, or is below the arbitrary 70 per cent minimum standard which experience has shown to be essential for the successful continuation of the course, they are asked to correct their errors, re-read certain portions of "the explanations" (or get direct teacher aid) and then, when they feel confident their difficulties are overcome, they may apply for a re-test on the unit. The tests are so constructed that theoretically the same test sheet can be used an infinite number of times without duplication or repetition of identical items.

Sample Test Item. An automobile radiator contains (8) qts. of which (10) % is (water; alcohol). How much alcohol; water) must be added to make it a (15) % solution?

Notice that when any suitable numbers are inserted in the parentheses, and certain key words are underscored, many different variations can be produced. Through this scheme pupils are taught to *generalize problem solving*—and *think relationally.*

A Provision for Socialized Procedures

It is firmly believed that complete individualization is not desirable, even if it were possible. Experience has shown that one period out of five (or its equivalent) should be used for general class discussion of concepts, methods of proof as applied to everyday problems of life, generalized problem-solving procedures, the influence of mathematics on the race, interesting recreations in mathematics, lives of great men of mathematics, etc., etc.

Results of the Milne Plan

The value of any procedure should be judged, among other things, by the effect upon the student's achievement, and the growth and development of well adjusted personalities.

The Regents results inevitably become the criterion for judging the achievement of the "Milne Plan" procedures. It is a criterion by which the pupils of New York are rated. I do not defend the situation nor denounce it—I merely state a fact.

In the Milne School the classes are, in the main, taught by cadet-teachers (under careful supervision).

All pupils who are regularly enrolled in the course are eligible for the Regents. None are barred from these examinations. In a period of five years no pupils have failed the Regents after completing our course. Some have demonstrated their ability to pass the Regents even though they had completed only two-thirds of *our* course.

Under these conditions the median grades on the Intermediate Algebra tests over this five-year period were extremely high. The medians ranged from *90* to *94*.

Last year a 9th grade boy finished all of the regularly prescribed elementary algebra course in one semester and all of another full year course in Intermediate Algebra during the second semester, and passed the Regents examination with a grade of *98*. Thus, two years of work were completed in one year. It might be well to state that this boy never attended a single class in Intermediate Algebra. All the work, including the tests, was done outside of the regular classroom—in study periods, etc. He refused to attend a class in which the pupils were two years older than himself. He was timid and self-conscious about it. The flexibility in this system of individualization permitted this adjustment to be made quickly and easily without any administrative complications.

Other boys and girls have finished the year's course in Intermediate Algebra in one semester, and passed the Regents with grades of *85* to *100*. As an experiment, they took the Advanced Algebra test (unofficially) and passed it without knowing that the previous course was designed for stimulating and extending our potential leaders to the limits of their individual capacities.

As they come back from college they point out that their first year college mathematics course is a pleasant, though wasteful, duplication of their "enriched course" in the Milne School.

Perhaps we need to extend these individualized procedures to the college level of instruction. Perhaps the Army and Navy types of Accelerated Programs have a kernel of good in them—*if the proper adjustments are made so that similar good features definitely outweigh the objectionable ones.*

Applications of Quadratic Equations

By William A. Cordrey

The advent of quadratic equations antedates the dawn of the Christian era by about two millennia. The study of conics, however, did not get under way until the fourth century prior to the birth of Christ. The first writer on this subject, Menaechmus, used the parabola and hyperbola in duplicating the cube. Several years later Euclid wrote a treatise on conic sections. This work was continued by Apollonius, whose investigations added much to the existing knowledge of the subject.

Several centuries elapsed after the methods and concepts peculiar to quadratic equations were evolved before they were used extensively by scientists. Investigations in various scientific fields gradually precipitated the conclusion that numerous natural laws conform to mathematical relations. Of these mathematical relations quadratic equations occupy a place of importance.

In the study of bodies falling from rest, the student used the equations,

$$s=\tfrac{1}{2}gt^2,\ v^2=2gs \quad ,$$

where s is the distance fallen, v the velocity at time t and g the gravitational constant equal to approximately thirty-two feet per second per second. These physical relation may be written thus:

$$t^2=(1/16)s,\ v^2=64s.$$

Both parabolas pass through the origin and are symmetrical with respect to the s-axis, the other axis in the first case being t and in the second case v. If the vertical axis be taken as the s-axis and the horizontal axis as both the t-axis and v-axis, it will be observed that the two parabolas have only the origin in common.

The slope of the first curve is given by $ds/dt=gt$, which is a linear relation between the slope and time, g being the proportionality constant. The rate of change of the slope, which is the second derivative, is equal to g. The slope of the second parabola is v/g and the rate of change of its slope is equal to the reciprocal of g. Thus, if the same distance is used to represent one unit on the v-axis as is used to represent a unit on the t-axis, the slope of the first parabola is the square of the acceleration times the slope of the second.

If the effect of air resistance be neglected, the first curve may be used to represent the relation between time and distance pertaining to an object thrown vertically into the air, considering time zero when the object comes to rest just before retracing its path. Then, the left half of the parabola represents the relation between time and distance while the object was ascending, and the right half represents the relation between the two variables during the descent of the same object over the same path. If the coefficient of s is reduced from 1/16, the arms of the parabola approach the positive s-axis. As the fraction decreases, the equation represents an object moving under a greater accelerating force and, hence, the distance traveled in a second is greater. As this fraction approaches zero, the acceleration and the velocity approach infinity and, consequently, the distance traveled becomes infinite while the time is very small.

The second equation presents a relation between velocity and distance. Time enters in only as it is a part of the definition of velocity. If we replace 64 by some parameter, p, and increase the value of the parameter, the parabola is shifted toward the v-axis. This means that at a given distance from the origin the object is traveling at a faster rate. As p approaches infinity, the increase in velocity approaches infinity and the path of the curve approaches coincidence with the v-axis.

If an object starts with an initial velocity of v, use may be made of the parabolic relation

$$s=vt+\tfrac{1}{2}gt^2$$

By translating the axes, an equation similar to one discussed above can be obtained with reference to the new axes.

If air resistance be neglected, the path of a moving projectile discharged with an initial velocity v at an angle A above the horizontal is given by

$$x=v\cos A\cdot t$$

$$y=v\sin A\cdot t-\tfrac{1}{2}gt^2,$$

where x represents the horizontal component of the distance traversed and y the vertical component. These relations describing the flight of a projectile are called parametric equations, t being the parameter. The elimination of t between the two relations will give

$$(v^2\cos^2 A)y=(\tfrac{1}{2}v^2\sin 2A)x-\tfrac{1}{2}gx^2.$$

This equation is quadratic in x and linear in y, and its trajectory is a parabola whose axis extends vertically downward and whose vertex represents the highest point attained by the projectile.

In case the object is discharged vertically upward, the equations are found by substituting 90 degrees for A in the parametric relations. They then become

$$x=0$$

$$y=vt-\tfrac{1}{2}gt^2.$$

The minus sign is the result of the effect of gravity being exactly opposite in direction to the effect of the discharging force. At the instant of discharge, t is zero and the two terms constituting the right side of the second equation are each zero. When $\tfrac{1}{2}gt^2$ becomes equal to vt, the value of y again becomes zero. The vertical velocity of the projectile is obtained by differentiating with respect to t and it may be given by

$$dy/dt=v-gt.$$

At the highest point gt equals v and the velocity of the projectile is zero. When the projectile first contacts the ground on its return fall, its velocity is the same as its initial velocity.

If the projectile is discharged horizontally, the equations are obtained by substituting zero for A. This gives

$$x=vt$$

$$y=-\tfrac{1}{2}gt^2.$$

The vertical velocity is downward and is entirely the result of gravity. These parametric equations show that the horizontal velocity at any point is unaffected by the pull of the earth. Likewise, the downward velocity which results from the gravitational pull is independent of the horizontal velocity of the discharged projectile. Hence, neglecting the effect of the atmosphere and the curvature of the earth, the projectile discharged horizontally will strike the earth at the same instant it would had it been dropped from the muzzle of the gun at the time it left the muzzle in a horizontal direction.

The kinetic energy of any moving object is given by

$$E=\tfrac{1}{2}mv^2,$$

where m is the mass of the object and v its velocity. This equation may be written

$$v^2=(2/m)E.$$

The length of the latus rectum is given by the coefficient of E. Hence, for heavier objects the latus rectum is smaller and the energy is larger in comparison with the velocity. The fact that the curve is symmetrical with respect to the E-axis merely means that the direction of the velocity has no effect on the amount of kinetic energy of the moving object. If the E-axis is taken as the vertical axis, the slope is the product of mass and velocity, and the rate of change of the slope is equal to the mass of the moving object. Hence, if the pound is chosen as the unit of weight and the foot as the unit of length, the slope of a curve is much less than it would be had gram and centimeter been used as the units in solving the same problem. Regardless of which system of units is used,

the graph representing the relation between the kinetic energy and velocity of a moving molecule will differ widely from the curve showing the relation between the same properties of a moving automobile. The first curve will have the smaller slope at every point with the exception of the origin. Similarly, the rate of change of the slope of a curve representing a rapidly moving train will be greater than that of an automobile; and when the equation is applied to a heavy star, the parabola all but coincides with the positive E-axis. Thus, as one graphs the kinetic energy equation for objects ranging from the mass of the lightest atom to the heaviest star, practically the entire area of two quadrants is used.

The centripetal force of a rotating object is given by

$$F=mv^2/r,$$

in which m represents the mass, v the linear velocity, and r the radius of the path of rotation. If r is constant, the curve showing the relation between F and v is a parabola passing through the origin and symmetrical with respect to the F-axis. When the value of r is increased, the arms of the parabola recede from the positive F-axis. Thus, in order to maintain a constant centripetal force while the radius of rotation is increased, it is necessary to increase the speed. In the case of a planet moving around the sun, the force of attraction between the two objects is very large and consequently the velocity must be great. The direction of the velocity has no effect on the centripetal force, as is shown by the fact that v occurs only to the second power in the equation.

This same relation may be used in the study of centrifugal action. If an automobile is moving in a circular path, the force tending to cause it to move away from the center of rotation is directly proportional to the velocity squared, and inversely proportional to the radius of the circular path. Consequently, as the curvature of the road decreases, it can be traversed at a higher speed. Highways are banked on the outside of the curve to counteract the force tending to cause the speeding automobile to move away from the center of rotation.

The vibration frequency of a simple pendulum conforms to the parabolic equation,

$$t^2=(4\pi^2/g)L,$$

in which t is the time of one vibration, L the length of the pendulum, and g the acceleration of a freely falling body. The value of g changes little as one goes from sea level to the highest mountain or from the equator to the poles. Still, the value of g does increase with a decrease in altitude or an increase in latitude, and the area contained in the parabola becomes slightly less. If it were possible to transfer the pendulum to the different heavenly bodies and the parabolas drawn, a much wider variation in the slopes of the curves would be observed, since the gravitational constant is different for different celestial bodies.

All of the numerous parabolic relations found in nature may be transformed into the equation,

$$x^2=4py,$$

where y and x are the ordinate and abscissa, respectively, and p the distance from the origin to the focus. The directrix is given by $y=-p$, and the length of the latus rectum by the coefficient of y. This curve passes through the origin and is symmetrical with respect to the y-axis.

A knowledge of the parabola is essential to the fabrication of the mirror of the reflecting telescope, and to the construction of headlights for automobiles, and of reflectors for searchlights. The creation of the mass spectrograph, used in the discovery of isotopes, presupposed an understanding of the parabolic relation. An appreciation of this simple equation has been of value in numerous inventions. In a number of cases both the parabola and the hyperbola have been used in the same scientific investigations.

Boyle's Law is a good example of the equilateral hyperbola. The equation is

$$PV=C,$$

where P represents pressure on an inclosed gas, V its volume, and C a constant. Experimental data give only one branch of the hyperbola since minus pressure or minus volume is impossible. However, the branch of the hyperbola falling in the third quadrant can be mathematically drawn. One variable must approach zero when the absolute value of the other one increases without limits and, consequently, the axes are the asymptotes.

With a constant volume, the pressure of a given quantity of gas is directly proportional to the absolute temperature; or, if the pressure is kept constant, the volume of a given mass of gas varies with the absolute temperature. Thus, with an increase in temperature there is an increase in C and when the temperature decreases, the value of C is diminished. The foci of the hyperbola recede from the origin as the parameter, C, is increased, and approach the origin as C is decreased. As C approaches zero, the hyperbola approaches the axes; and when the constant becomes zero, the graph is the axes.

If the values of P are taken as ordinates, the slope of the curve is given by $dP/dV = -C/V^2$. Since C is a positive constant and since V occurs only to the second power, the slope is always negative. It increases in absolute value as the numerical value of V decreases, and it decreases in absolute value as the numerical value of V increases. Thus, the slope approaches minus infinity as V approaches zero, and it approaches zero as the absolute value of V increases without limit. A similar relation can be given for the slope of any equilateral hyperbola.

If the density, D, be substituted in the above equation for P, the constant is the mass, M, of the gas and the relation,

$$DV = M,$$

is still hyperbolic. Again, the branch of the hyperbola in the third quadrant cannot be derived from scientific data, but it can be mathematically drawn. Theoretically it is possible to make M a parameter by adding to or subtracting from the given quantity of gas. As M increases, the foci of the hyperbola move farther from the origin, always having the same asymptotes and always being symmetrical with respect to the origin and the line bisecting the first and third quadrants.

The magnifying power of a telescope is given by

$$fm = F,$$

in which F is the focal length of the objective and f the focal length of the eye-piece. If numerous eye-pieces be used with a given objective, the relation is a hyperbolic one. Hence, with a given objective, the magnification of the telescope increases as the focal length of the eye-piece decreases. Likewise, the value of m decreases as the value of f increases, and the magnifying power is unity when f is equal fo F.

If objectives with different focal lengths are substituted for the original objective, F becomes a parameter and consequently the equation is represented by different hyperbolas. These curves recede from the origin as F increases and approach the origin as F approaches zero, the relation always being a hyperbolic one.

Ohm's Law states that the electromotive force in volts is equal to the product of the current in amperes and the resistance in ohms. This is given mathematically by

$$IR = E,$$

where I denotes the current, R the resistance, and E the electromotive force. If a large number of resistances be connected in parallel so that the voltage drop is the same across each resistance, the relation between the current and resistance will be hyperbolic. Data can be obtained by placing a rheostat in series with an ammeter. As the reading of the rheostat is decreased, the ammeter indicates a larger current. Similarly, as the value of the resistance is increased, the reading of the ammeter is reduced such as to keep the product of the two readings constant.

The data obtained from the meter and

the rheostat will give only one branch of the hyperbola. The other branch can be obtained by reversing the direction of the current. In case an alternating current is used, the entire hyperbola is obtained at one time. The value of the parameter, E, can be increased or decreased by changing the number of batteries if a direct current is used or by the use of transformers if an alternating current is used.

Only a few of the large number of natural laws conforming to the hyperbolic relation have been mentioned. Several others could be added. The product of the wave-length of the maximum point of a black-body radiation curve and the absolute temperature of the object producing the radiation is a constant. Research pertaining to nuclear scattering has shown that particles execute hyperbolic orbits under certain conditions. Under other conditions, elliptical paths are followed. The latter curve is frequently encountered in the study of the motions of heavenly bodies.

Kepler's first law states that the orbit of each planet is an ellipse with the sun at one focus. Thus, the path of a planet is given by

$$x^2/a^2+y^2/b^2=1,$$

where a and b are the major and minor semi-axes. The intersection of the axes is at the center of the ellipse and, consequently, the curve is symmetrical with respect to the origin. If the x-axis is the major axis, the foci are at $(c, 0)$ and $(-c, 0)$, where c is the square root of the difference of the squares of the semi-axes. The ratio of the distance of the focus from the origin to the semi-major axis is called the eccentricity. The orbits of the planets have eccentricities varying from 0.206 for Mercury to 0.007 for Venus. These orbits do not all lie in the same plane, but they do have the center of the sun as a common focus.

The joint action of two simple periodic motions of equal frequencies and having displacements in two perpendicular directions is given by

$$x^2/a^2+y^2/b^2-(2xy/ab)\cos A=\sin^2 A,$$

where A is the phase difference and a and b are constants. The curves represented by this relation are ellipses with the exception of a few special cases where they degenerate into straight lines. These special cases can be examined by substituting integral multiples of π for A. When A is equal to an even integer times π, the equation becomes

$$y=(b/a)x,$$

which represents a line having the slope given by the coefficient of x. In the cases where A is the product of π and an odd integer, the equation of the line is

$$y=-(b/a)x,$$

in which the slope is numerically the same as in the preceding relation, but opposite in sign. All of these lines pass through the origin.

The major and minor axes of the ellipse coincide with the axes of the coordinates when A is the product of an odd integer and a right angle. When this condition exists, the equation is

$$x^2/a^2+y^2/b^2=1,$$

which is the general equation of an ellipse with the center at the origin and symmetrical with respect to both axes. If a and b are made to approach the same quantity, the curve will approach a circle which is a special case of an ellipse.

When a pebble is thrown into a pond of still water, the transverse wave travels outward from the original point of disturbance with an equal velocity in all horizontal directions. The position of the wave front at any time, t, is given by

$$x^2+y^2=s^2,$$

where x and y are the coordinate axes passing through the point of origin of the wave and extending horizontally, and s the distance from the origin to the wave front. The value of s is equal to the product of the velocity and the time. Similarly, the cross-section of the wave front of a light wave is given by this equation, where the

axes lie in the cross-sectional plane with their intersection at the point of intersection of the plane and the normal to the plane passing through the point of origin of the wave.

Numerous natural laws can be expressed in the form of quadratic equations and represented by curves known as conic sections. These relations and curves were investigated by early mathematicians, and new material and new methods have been added by subsequent scholars in the same field. This useful knowledge, made available by mathematicians, was held in readiness for the scientists who could use it in solving the mysteries of nature.

Mathematical Songs

Tune: *Battle Hymn of the Republic*

1. Oh we're the Mathematics Club
Of known propensity
We're pursuing mathematics with
A great intensity.
'Tis a very living subject of
A great fertility.
Its truth is marching on.

2. A universal language just
The same the wide world o'er
It started from such simple things
As two and two make four.
And every day in every way
Its growing more and more.
Its truth is marching on.

3. Its multiplied and multiplied
And multiplied again.
A genius wouldn't learn it all
In three score years and ten.
We mean to add our little bit
To make it grow again.
Its truth is marching on.

4. The key that keeps the planets
Whirling blithely round the sun
If they tripped on an equation, then
Our race would soon be run.
The language of the gods, it is
The universal tongue.
Its truth is marching on.

Chorus

Glory, Glory to our Subject
Glory, Glory to our Subject
Glory, Glory to our Subject
Its truth is marching on.

Tune: *The Wearing of the Green*

1. Oh X how I do loathe you
You slippery old unknown,
I'm sure you hold the record
For the sorrows you have sown.
I seek you in the darkness
That forms my mental night,
And when I think I have you
You vanish from my sight.

2. If this is this and that is that
Pray what is so and so?
The teacher asks me every day
I'm sure I never know.
She thinks I'm very stupid
But I am not to blame.
It's you, you skulking creature who
Won't answer to your name.

3. Most valiantly I hunt you out
In every likely lair,
But when I catch a glimpse of you
You wink and disappear.
With wings upon your shoulders
And heels awinged too,
Alas, can I, poor mortal,
Expect to cope with you.

4. Your heart is full of mischief
Brewing troubles by the peck,
If ever I do corner you
Be sure I'll wring your neck.
But X I want to warn you
You really must beware,
Or you'll never get to Heaven
For there are no secrets there.

A Mathematics Work Room for the Senior High School

By Jessie Roselle Smith
Technical High School, Saint Cloud, Minnesota

In the field of mathematics, the advent of the war has not yet created many major changes. The most noticeable one is that more pupils are now enrolled in the mathematics courses than formerly. That the pupils are learning the subject better than in pre-war days is also quite apparent. I am not quite so certain that we are teaching it better. The war is the cause of the great increase in the enrollments, and the pupil's motive for studying mathematics is the most powerful in existence, that of retaining his life, and the life of freedom. If we do not recognize the strength of this force acting in our favor there is danger of our becoming apathetic. It behooves us to add vitality and meaning to the abstract theory that we teach. Unless we improve the quality of our work, and arouse the interest of the pupils in this field, we are going to experience a terrific jolt when the motive furnished by the war is gone.

The selection of the best method to be employed in the teaching of mathematics is still in the controversial stages. The exclusive use of the traditional method too often results in the thwarting of many of the most desirable objectives of education. When we resort to the use of any of the more modern teaching procedures, we usually discover that the pupil has failed to master the fundamentals of the principles of mathematics. The plan that I am proposing is in the nature of a combination of the ingredients of these allegedly conflicting educational programs. The presentation of the subject matter in the approved traditional manner is complemented by facilities in a work room for investigational techniques and the acquisition of desirable mental attitudes.

In an effort to arouse in pupils that very desirable, though elusive, quality of initiative, we organized what we call *The Mathematics Work Room* in our senior high school. The project has been fairly successful in attaining this result, and incidentally many others. Provision is made here for individual differences. For pupils of unusual talent, materials that stimulate unusual qualities of imagination, originality, and initiative are available; for others, a supply of material that offers a real challenge is supplied. In this room the student selects his own projects, thus fostering the acquisition of an attitude of inquiry. Working in groups as they so often do, the proper social habits are formed. Here too, a functional knowledge of the use of instruments is gained. The pupils enjoy the application of principles learned, and derive much satisfaction from exploring, experimenting and performing tasks not assigned by the teacher. One definite result of the use of this room is the great increase in the enrollment of pupils in subsequent courses.

In order to reap such results much time and thought must be put into preparation of such a place. An unused class room or a part of the mathematics recitation room can be used for this purpose. Our *Mathematics Work Room* is open at all times to all pupils in the school. Many prefer to spend their study period there and prepare their text-book assignments at home. There is no one in charge of the room. All that we ask is that each register when entering it, and record how he has been occupied when leaving.

The work room contains several tables, the most popular of which is the one designated "Army and Navy." On this is placed, in addition to the usual material sent to the schools for inductee perusal,

various books containing problems in special military fields, sample questions used in West Point and Annapolis entrance examinations, aviation quiz books, a portfolio of war pictures from *Life* magazine, and much helpful material sent to us by the boys already in service. The bulletin board for this table contains clippings from the local newspaper on activities of the men in service who have taken mathematics at our high school. Comments are often made on the fact that these men invariably become commissioned officers.

I shall not attempt to enumerate the many perplexing attractions found in the "Tricks and Puzzle" corner. This is the feature that entices many of the non-mathematical people to the Work Room. From there, they usually wander to the more worth-while tables, one of which is "Recreational Reading." I would like to caution any one who is planning on having such a room against placing magazines on this table, as there is such a wealth of mathematical material from which to draw. We have there, besides the many good books on mathematics, a portfolio of good mathematical pictures, booklets of puzzles collected and compiled by pupils, several of the Blue Book series, material on lives of great mathematicians, information on vocations, a collection of cartoons and jokes of a mathematical nature, the *Pocket Entertainer*, *The Riddle Book*, *Curiosities of Mathematics*. Here too is a "Correspondence Folder," containing testimonials from former mathematics students on the use that they have made of certain items studied in mathematics in the various occupations in which they are engaged.

It is the "Work Tables" that most justify the existence of this room and that require the most preliminary preparation on the part of the teacher. In the racks on these tables there are several portfolios with suggestions for procedure written on the cover of each. The secret of the success of this feature is that these suggestions appeal to the pupils. To us, the generalization graph is a reminder of the investigations preliminary to the writing of a master's thesis. To them, it is a simple and revealing process if they act on any of the following directions on the portfolio on Generalization Graphs:

> If you examine the contents of this folder you will see the results of some investigations made by former pupils.
>
> If *you* have a question in your mind as to the cause or effect of some action, take data for a month or more on both. At the end of that time, graph the data. You may arrive at some startling discoveries that are new to mankind. This is good training for your mind—observe, record, generalize.
>
> Do not waste your time making bar graphs unless you wish to convince some one of the marked relation between several things. The making of bar graphs takes much time, but requires little mental effort. That is why they are so prevalent.

The suggestions on the portfolio on Instruments for the Measurement of Angles, are as follows:

> Practice measuring angles with our transits. Will you please leave in this folder the results of your measurements, and a description or diagram of the points measured.
>
> Boys who have had training in wood work might like to make another transit for their own use. Directions for making one are contained in this folder.
>
> Try measuring the diameter of the wire, to a high degree of accuracy by the use of the micrometer. Record your results and compare with the results of others.
>
> Measure the angle of elevation of some high object such as the top of the flag pole, the sun, etc., by use of the sextant. Always leave in folder a record of your measurements.
>
> It would be interesting to measure the angle of elevation of the sun at the same hour for several days and then graph the results. Do you think that the graph would be a straight line?
>
> They say that "Mathematics has been taken aloft." This means that measurements will often be taken from a plane, so include in your use of the sextant, the measurement of angles of depression.

The directions on the cover of the portfolio on Harder Mathematics are:

> Here are some of the questions that people often ask me. Perhaps you would like to investigate one of these and then write a simple explanation that would be comprehensible to most people. Then when questioned, I could merely refer them to your paper. Remember that giving an example will often make clear a difficult idea. You can get information for some of these from pamphlets in the folder and from books in this room.

1. How is a table of logarithms made?
2. How can I read a large number of many digits?
3. Can you give me directions for making my own pantograph?
4. How can I construct an ellipse of given length and width with string and tacks?
5. How is a slide rule made?
6. How is *pi* obtained? What is its exact value?
7. I would like to see a set of simultaneous equations with four unknowns, solved by determinants.
8. What is the fourth dimension?
9. What makes an airplane fly?

Other portfolios on the Work Tables, each with helpful directions for procedure are: Crossword Puzzles, Air Maps, Magic Squares, Mazes, Geometric Designs, Slide Rules, Instruments, Projects, Navigation. The planning of these takes much of the teacher's time but after that preliminary work is done the pupil's activity follows automatically.

The number of articles produced is astonishing. Among others are air maps with Saint Cloud as the center, a sun-dial for places of 46° north latitude, contour maps of local regions, designs for airplane wings, plan for a flower garden in the form of a maze, written specifications for a city water system, mathematical crossword puzzles, slide rules, transits, polyhedron Christmas tree decorations, models of solid geometry theorems made from balsam wood strips or wire. The work here, as in all fields, varies directly as the interest aroused. The enthusiasm developed in the Work Room motivates to high degree the interest in the class room, and establishes in the pupil's mind a receptive attitude there. It is gratifying too to note how quickly the students become mathematics conscious in such an environment, and bring contributions from their homes to this room.

It has ever been my experience that no matter how well a skill is taught it does not become a part of the pupil until he has repeated it, or used it at some later time. This constant reviewing is tedious to both teacher and pupil. For this reason I consider the most successful of all our adventures in this room that one which we call, The Problem of the Week. Many copies of one interesting problem are placed on the entrance table every Monday morning. Each takes a copy of it as he enters the room. If successful in solving it he slips it into the drawer of the desk, if not, he replaces it for others to use. At the end of the week I post the names of the correct solvers and a correct solution. To obtain favorable results, the problems selected must appeal to the boys and girls. Here are a few that I have used:

1. *A bombing plane is dispatched to an objective 600 miles from its base. There is a head wind of 15 miles per hour on the way over. Having dropped his bombs, the pilot is able to increase his speed by 10 miles per hour, and makes the return trip, with the wind on his tail, in a half hour less time. What is the ground rate of the plane?*

2. *The rat is said to be a prolific little creature. Most will start breeding at the age of three months. From then on they'll produce on the average six litters a year, with an average of ten rats a litter. Can you figure out how many rats one pair could be responsible for, by interbreeding, if they lived two and one-half years? This thought is even more staggering when told that it takes around two cents a day to feed a rat.*

3. *A lookout stationed in the "crow's nest" of a battleship at a height of 100 feet above the water is ordered to scan the surrounding water for enemy periscopes. How far could he see on a clear calm day?*

4. *The toughest problem that Jackie in the Quizz Kids Program ever solved mentally is so complicated that just to read it may make you dizzy. Here it is: A circle is 8 inches in diameter. Inside the circle is inscribed an equilateral triangle, and inside triangle another circle, and so on until there are five circles. All six-year old Jackie had to do was to figure out the area of the last circle. Can you do it?*

All of the pupils read these problems and most of them try to solve them. They arouse much pupil discussion, which extends to their homes and to members of

the faculty. The absorbing topic of conversation at a recent faculty tea, engaging the attention of every male member present, was how a square whose area is 64 sq. in. can be transformed into a rectangle whose area is 65 sq. in. The problem was introduced by the music master, who had not been able to sleep nights since one of the math boys had disturbed the serenity of his mind by propounding it to him.

The idea of a mathematics work room is not new. However, too many of them fail to function. Some are patterned after libraries. The material is placed on shelves or in cupboards, and is rarely used by the pupils. Others are patterned after science laboratories. In them, the pupils are not given enough freedom. There is already too much in our mathematics recitation that tends to destroy the child's power of independent thought and action. My plea is that in this work room he be permitted to derive the satisfaction attained from *freedom* in the selection of his tasks.

A Model for Napier's Rules

By William R. Ransom
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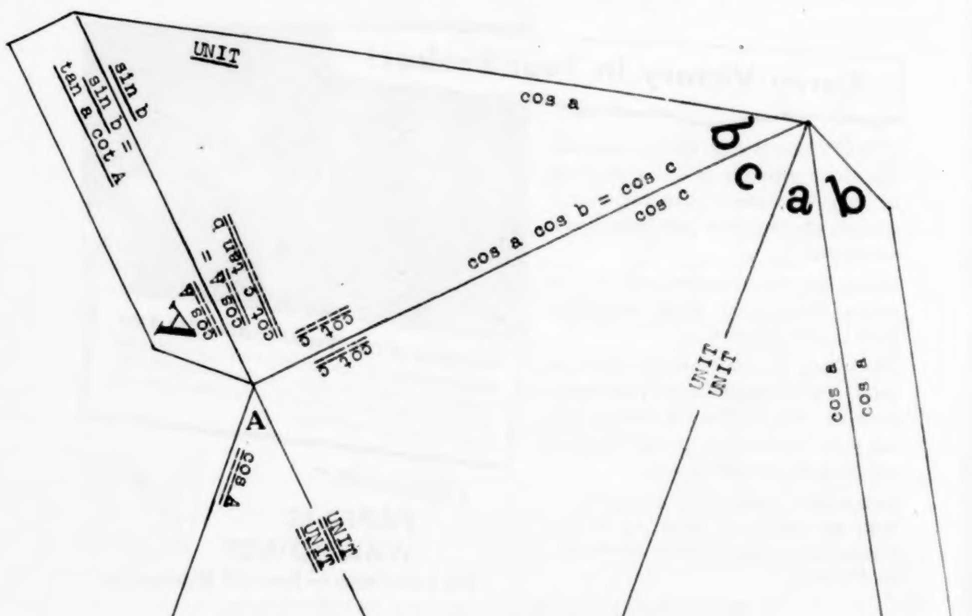
Students often naturally want to have the formulas for the right spherical triangle derived so as to come out directly in the forms in which Napier's Rules give them. The tetrahedron which results from folding this diagram presents such derivations for all except the three which contain both angles. (It is assumed that the angle A may be renamed B, with a corresponding interchange of a and b.) Is there any geometric proof for the formulas that contain both A and B? Using only Napier formulas they can be derived by multiplying three formulas, as below, and dividing out the sines and cosines that appear twice:

$$\begin{cases} \sin a = \tan b \cot B \\ \sin b = \tan a \cot A \\ \cos c = \cos a \cos b \end{cases} \qquad \begin{cases} \sin b = \sin B \sin c \\ \cos A = \tan b \cot c \\ \cos c = \cos a \cos b \end{cases}$$

whence

$$\cos c = \cot A \cot B \qquad \cos A = \sin B \cos a$$

On the model, each formula results from comparing the lengths of three lines that meet at a vertex. The underscoring system (colors would be better) indicates which edge is taken as the unit in each case.



◆ THE ART OF TEACHING ◆

Square Root by Approximation and Division

By William S. Tobey
Junior and Senior High Schools, Long Branch, N. J.

Recently several references have been made to the method of finding square roots by approximation and division. There appears to be a belief among some teachers that this method is longer than the conventional method. This has not been found true in actual tests given to both rapid and slow groups over a period of several years. Finding square roots by approximation and division has been in use fifteen years in the senior high school of Long Branch, New Jersey for two reasons. First, few pupils have come to the senior school during this period with the ability to find the square root of a number despite the fact that several weeks had been devoted to teaching it in an earlier grade. Second, to most pupils, the conventional method is one more example of "mechanical manipulation" and as such has no standing where learning is the goal.

To find the square root of 7 by the method of approximation and division the pupil must first be led to see that since 7 lies between 4 and 9 its square root must lie between 2 and 3. Then he must see that since the distance between 4 and 9 is 5 and that between 4 and 7 is 3 the square root must lie about $\frac{3}{5}$ the way from 2 to 3 or is about 2.6. In changing $\frac{3}{5}$ to a decimal the pupil is required to multiply both members by 2 or change the fraction $\frac{3}{5}$ to 6/10. The next step is to divide 7 by 2.6.

At this point the pupil is led to see that since the remainder 24 is nearly as large as 26 his approximation is too small. The quotient should be 2.7. He is then required to average 2.6 and 2.7 and obtain 2.65.

```
2.6|7   |2.6
    52
    ---
    180
    156
    ---
     24
```

The next step is to have the pupil divide 7 by 2.65.

The pupil discovers that the quotient 2.64 is less than the divisor 2.65 so the root must lie between 2.64 and 2.65.

```
2.65|7    |2.64
     530
     ----
     1700
     1590
     ----
      1100
      1060
      ----
        40
```

He is required to average 2.64 and 2.65 and divide 7 by this result.

Since the remainder 1330 is greater than half 2645 the quotient should be 2.647.

The number 2.646 which lies half way between 2.645 and 2.647 is the square root of 7 to the nearest thousandth.

```
2.645|7     |2.646
      5290
      -----
      17100
      15870
      -----
       12300
       10580
       -----
        17200
        15870
        -----
         1330
```

The objection may be raised by some that this method involves much work. With small numbers the dividing and averaging must be repeated several times but as the size of the number increases the approximation more closely approaches the true root.

In the case of a number such as 1127 the pupil must first be led to see that the 11 are hundreds so the square root must lie between 3 tens and 4 tens. His next step is to place 1127 between 900 and 1600.

```
1600
1127
 900
```

The difference between 900 and 1600 is 700 while that between 900 and 1127 is 227. The square root of 1127 is about 227/700 the way

from 30 to 40. At this point there is splendid opportunity for fine teaching since the procedure depends to a considerable degree on the nature of the numbers. In this case a good procedure is to inquire of the pupil how he would change 700 to 10. He can be led to see that he must divide by 70. Doing this to both members he finds that the numerator 227 becomes about 3. His approximate square root is 33. The next step is to square 33 obtaining 1089. This is smaller than 1127 so he will next square 34 obtaining 1156. He must then place 1127 between 1089 and 1156. Performing the following sub-

1156	1127	1156
1127	1089	1089
1089	——	——
	38	67

tractions the pupils discovers that 1127 is 38/67 the way from 1089 to 1156 so its square root is about 38/67 the way from 33 to 34. Since 67 is about $\frac{2}{3}$ of 100 it must be increased one-half itself to become 100. Increasing both members of 38/67 by one-half the fraction becomes very nearly 57/100. The square root of 1127 is about 33.57. The next step is to divide 1127 by 33.57.

No attention is paid to place of decimal point since the pupil knows that the square root is about 33.57.

Averaging 33.57 and 33.572 the result is 33.571 and is the square root of 1127 to the nearest thousandth.

```
33.57 | 1127    | 33.572
        10071
        -----
         11990
         10071
         -----
          19190
          16785
          -----
           24050
           23499
           -----
            5510
            6714
```

Every step by this method requires estimating and judging. Pupils can begin with small numbers in the seventh grade, proceed to large numbers in the eighth grade and in the ninth grade find the approximate square roots of large numbers without any dividing. It would probably be advantageous to the cause of arithmetic if many of the division exercises were eliminated and the necessary facility in the process gained in the seventh and eighth grades in connection with finding square roots.

In the tenth grade for those pupils who elect plane geometry the geometric interpretation would serve as a refresher of their work in the earlier grades and expand their understanding of the process.

In eleventh year algebra square root could be treated as the inverse of $(a+b)$ squared. In this case it would serve a double purpose. It would not only broaden their knowledge of the process but would in addition help to relieve their minds as to what happens to the $2ab$ when one takes the square root of $a^2+2ab+b^2$.

A procedure similar to the one here outlined would give our abler pupils a rich background in the matter of square root and provide the others with an understandable workable method.

Program of the Joint Metropolitan Meeting of the National Council of Teachers of Mathematics

Teachers College, Columbia University
525 West 120 St., New York City
Saturday, March 24, 1945

CO-OPERATING ORGANIZATIONS

Standing Committee on Mathematics in New York City
Dr. Eugenie C. Hausle, President
Association of Chairmen of Departments of Mathematics of New York City
Mr. Benjamin Braverman, President
Section 19 (Mathematics) of the New York Society for the Experimental Study of Education
Professor William L. Schaaf, President
The Association of Mathematics Teachers of New Jersey
Dr. J. Dwight Daugherty, President
Association of Teachers of Mathematics of New York City
Mr. Samuel Altwerger, President
Nassau County Association of Mathematics Teachers
Mr. Gerald Thayer, President
Suffolk County Association of Mathematics Teachers
Mrs. Harriet Burgie, President
Westchester County Mathematics Teachers Association
Mr. Rupert Smith, President
Theme: *Tomorrow's Mathematics*

MORNING SESSION

10:15–12:15

Panel I. *Functional Competency in Mathematics*
Horace Mann School Auditorium,
Corner 120th and Broadway

Presiding: Virgil Mallory, Montclair State Teachers College, N.J.; Member of the Policy Commission of the National Council.

Speakers:

1. What the High School Teachers Should Know about the Teaching of Arithmetic. Amanda Loughren, Supervisor of Mathematics, Public Schools, Elizabeth, N. J.
2. How the Junior High School Articulates with the Elementary School and the Senior High School. Mary Rogers, Head of Mathematics Department, Roosevelt Junior High School, Westfield, N. J.
3. Mathematics Courses in the Senior High School Differentiated According to Needs. Simon Berman, Chairman, Department of Mathematics, Stuyvesant High School, N. Y. C.
   Benjamin Braverman, Chairman, Department of Mathematics, Seward Park High School, N. Y. C.

Panel II. *Special Problems*

102 Dodge Hall

Presiding: William L. Schaaf, Brooklyn College, Brooklyn, N. Y.; Member of the Policy Commission of the National Council.

Speakers:

1. Mathematics for the Veteran. Ida Ostrander, Chairman, Department of Mathematics, Sewanhaka High School, Floral Park, N. Y.
2. The Training of Mathematics Teachers. Howard Fehr, State Teachers College, Montclair, N. Y.
3. Mathematics in the "Institutes" and in the Junior Colleges. John R. Clark, Teachers College, Columbia University, N. Y. C.

LUNCHEON

1:00 P.M.

Men's Faculty Club, Columbia University, Morningside Drive and 117 St., N. Y. C.

Presiding: Eugenie C. Hausle, Member of the Policy Commission of the National Council on Post-War Plans in Mathematics.

Toastmasters: William D. Reeve, Teachers College, Columbia University; Carl N. Shuster, State Teachers College, Trenton, N. J.

"The National Council Looks Ahead"

Rolland R. Smith, Guest Speaker: Co-ordinator of Mathematics, Springfield, Mass., Public Schools, and past president, National Council of Teachers of Mathematics.

COMMITTEE ON ARRANGEMENTS

Samuel Altwerger, Bronx Vocational High School, N. Y. C.
Benjamin Braverman, Seward Park High School, N. Y. C.
Eugenie C. Hausle, James Monroe High School, N. Y. C.
Nathan Lazar, Midwood High School, Brooklyn, N. Y
Kenneth Morgan, High School, Mt. Kisco, N. Y.
Max Peters, Long Island City High School, Long Island, N. Y.
Alice Reeve, High School, Rockville Centre, N. Y.
Mary Rogers, Junior High School, Westfield, N. J.
William L. Schaaf, Brooklyn College, Brooklyn, N. Y. (Chairman)

EDITORIALS

Sir Percy Nunn, Professor Emeritus of Education at the University of London, died in Madeira on December 12, 1944, according to a report in a recent issue of *The Schoolmaster and Woman Teacher's Chronicle* (London). Sir Percy, who was seventy-four years old at the time of his death, had served as a secondary schoolmaster from 1891 to 1905 when he was appointed vice-principal of the London Day Training College. In 1913, he was made Professor of Education in the University of London, a position which he held until his retirement in 1937. In 1922, he became the principal of the London Day Training College, and when (in 1932) the College became the Institute of Education of the University he continued in the directorship until 1936.

Sir Percy was perhaps most widely known for his book *Education, Its Data and First Principles*, a revision of which is to be published shortly. His contribution to the field of mathematics not only in England, but in the British Empire, was notable. His three-volume set of books the first on *The Teaching of Algebra* (*Including Trigonometry*) and the second and third on *Exercises in Algebra* (*Including Trigonometry*) have had a great influence even in this country, particularly among leaders in the field. His influence is being manifested now in current textbooks. His books contain great source material for all teachers of algebra.

Sir Percy lectured one summer at Teachers College and those of us who were fortunate enough to hear him will never forget the great spirit of the man and his love for mathematics. He anticipated much of the present emphasis upon the importance of the practical in mathematics and yet he did not neglect the cultural aspects. He said:

> Mathematical truths always have two sides or aspects. With the one they face and have contact with the world of outer realities lying in time and space. With the other they face and have relations with one another. Thus the fact that equiangular triangles have proportional sides enables me to determine by drawing or by calculation the height of an unscaleable mountain peak twenty miles away. This is the first or outer aspect of that particular mathematical truth. On the other hand I can deduce the truth itself with complete certainty from the assumed properties of congruent triangles. This is its second or inner aspect. The history of mathematics is a tale of ever-widening development on both these sides. From its dim beginnings by the Euphrates and the Nile mathematics has been on the one hand a means by which man has constantly increased his understanding of his environment and his power of manipulating it, and on the other hand a body of pure ideas, slowly growing and consolidating into a noble rational structure. Progress has brought about, and, indeed, has required, division of labour. A Lagrange or a Clerk Maxwell is chiefly concerned to enlarge the outer dominion of mathematics over matter; a Gauss or a Cantor seeks rather to perfect and extend the inner realm of order among mathematical ideas themselves. But these different currents of progress must not be thought of as independent streams. One never has existed and probably never will exist apart from the other. The view that they represent wholly distinct forms of intellectual activity is partial, unhistorical, and unphilosophical. A more serious charge against it is that it has produced an infinite amount of harm in the teaching of mathematics.
>
> Our purpose in teaching mathematics in school should be to enable the pupil to realize, at least in an elementary way, this two-fold significance of mathematical progress. A person, to be really "educated," should have been taught the importance of mathematics as an instrument of material conquests and of social organization, and should be able to appreciate the value and significance of an ordered system of mathematical ideas. There is no need to add that mathematical instruction should also aim at "disciplining his mind" or giving him "mental training." So far as the ideals intended by these phrases are sound they are comprehended in the wider purpose already stated. Nor should we add a clause to safeguard the interests of those who are to enter the mathematical professions. The treatment of the subject prescribed by our principle is precisely the one which best supplies their special needs.*

Let us hope that the future Nunns in the teaching field will be numerous.

W.D.R.

* Nunn, T. Percy. *The Teaching of Algebra* (*Including Trigonometry*), Longmans, Green and Co., London, 1931, pp. 16–17.

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS has a right to be proud of its series of Yearbooks (18 in all to date). This series compares favorably with any other existing series of Yearbooks prepared in any other educational field. This much can be said without boasting, because the facts will bear out the statement. Moreover, the influence of the Yearbooks over the years is beginning to be felt not only in the discussions, but to some extent at least in textbooks and magazine articles. However, it is clear that many libraries and teachers have not availed themselves of the opportunity to obtain these books. The supply of several of the Yearbooks is now running low. The first, second, and tenth Yearbooks are now completely sold out and we have only a small number of most of the others. These Yearbooks will not be reprinted; so those who do not have copies should order them at once from The Bureau of Publications, Teachers College, 525 West 120 Street, New York 27, N. Y. Yearbooks 3 to 16 inclusive, except the 10th, are each $1.75 postpaid. Yearbooks 17 and 18 are $2.00 each postpaid.

As an example of what some teachers are missing take the third Yearbook on *Selected Topics in the Teaching of Mathematics*. Where can a mathematics teacher go to get more for his money? Here is the Table of Contents.

It is to be hoped that the State Representatives of the Council and members of the Council generally will do all they can to advertise the Yearbooks in their respective localities.

W.D.R.

Miss Ina E. Holroyd has retired as editor of the Bulletin of the Kansas Association of Teachers of Mathematics after many years of devoted service. It is teachers like Miss Holroyd whose labor of love in the cause of mathematics encourages others of us to carry on in spite of discouragements due to the apathy and lack of interest of many teachers beyond their narrow spheres of interest. We trust that Miss Holroyd will now be able to give much of her time and zeal to a larger area. THE MATHEMATICS TEACHER congratulates Miss Holroyd on her past achievement and wishes her many more years of service in the cause of better teaching of mathematics.

W.D.R.

IN OTHER PERIODICALS

By Nathan Lazar
Midwood High School, Brooklyn 10, New York

The American Mathematical Monthly

November 1944, vol. 51, no. 9.

1. Halmos, P. R., "The Foundations of Probability," pp. 493–510.
2. Min, Szu-Hoa, "Non-Analytic Functions," pp. 510–517.
3. War Information: The Mathematical Tables Project; New Armed Forces Institute Catalogue; Disposal of Surplus War Property; Mathematicians Needed in Federal Office. Pp. 540–544.

Bulletin of the Association of Mathematics Teachers of New Jersey

February 1945.

1. Sinton, Elizabeth F., "Report of the Eightieth Regular Meeting," pp. 1–3.
2. Daugherty, J. Dwight, "The President's Message and Announcement of Research Committees," pp. 3–5.
3. Mallory, Virgil S., "Post-War Commissions," pp. 6–9.
4. Fehr, Howard F., "Scholarship and the High School Teacher," (Editorial), p. 10.
5. Rogers, Mary C., "Membership Message," pp. 11–14.
6. Chertoff, Isadore, "A Suggested Program for a High School Mathematics Club Meeting," pp. 15–18.

Bulletin of the Association of Teachers of Mathematics of New York City

January 1945, vol. 1, no. 1.

1. Editorials, p. 1.
2. Altwerger, Samuel I., "In Search of a Panacea," pp. 2–3.
3. Gordon, David, "An Historical Introduction to the Theory of Limits," pp. 3–7.
4. Sitomer, Harry, "The Gestalt in Teaching Plane Geometry," pp. 8–9.
5. "The Problem Page," pp. 11–12. (Edited by Mannis Charosh).
6. Wayne, Alan, "The Problems of Junior High School Mathematics Teaching," pp. 13–14.
7. "News Notes," pp. 14–15.

School Science and Mathematics

January 1945, vol. 45, no. 1.

1. Sleight, Norma, "Arches Through the Ages,—Applications to Geometry," pp. 21–25.
2. Engelhart, Max D., "The Role of the Science or Mathematics Teacher in Meeting the needs of the Veteran Who Returns to School," pp. 27–32.
3. Jerbert, A. R., "The Algebraic Number Scale," pp. 40–44.
4. Murray, Walter I., "The Problem of Reading in Mathematics," pp. 54–61.
5. Douglass, Harl R., "Adapting Instruction in Science and Mathematics to Post-War Conditions and Needs," pp. 62–78.
6. Nyberg, Joseph A., "Notes from a Mathematic Classroom," pp. 83–87.

Miscellaneous

1. Baxter, B., "Forward-Looking Practices in Arithmetic," *California Elementary School Principals Association, Sixteenth Yearbook*, pp. 97–107.
2. Brueckner, L. J., "Synthesis of Scientific Studies," *Journal of Educational Research*, 38. 146–148, October, 1944.
3. Buswell, G. T., "Selected References on Elementary-School Instruction. Arithmetic," *Elementary School Journal*, 45. 164–165, November, 1944.
4. Collyer, C., "Another Side of Mathematics," *School* (Secondary Edition), 33. 240–242, November 1944.
5. Curtis, F. D., "Mathematical Vocabulary Used in Secondary-School Textbooks of Science," *Journal of Educational Research*, 38. 124–131, October, 1944.
6. De May, A. J., "First Steps in Arithmetic," *Instructor*, 53. 28–29+, October, 1944.
7. Greenfield, S. C., "Whole Is Greater Than Sum of Its Parts," *High Points*, 26. 78–79, November, 1944.
8. Hansen, C. W., "Factors Associated with Successful Achievement in Problem Solving in Sixth Grade Arithmetic," *Journal of Educational Research*, 38. 111–118, October, 1944.
9. Johns, E., "Blackboard Goes to War," *Business Education World*, 25. 120–121, November, 1944.
10. Johnson, H. C., "Effect of Instruction in Mathematical Vocabulary Upon Problem Solving in Arithmetic," *Journal of Educational Research*, 38. 97–110, October, 1944.
11. MacLatchy, J. H., "Seeing and Understanding in Number," *Elementary School Journal*, 45. 144–152, November, 1944.
12. Murnaghan, F. D., "On the Teaching of Mathematics," *Science*, 100. 479–486, December 1, 1944.
13. Treacy, J. P., "Relationship of Reading Skills to the Ability to Solve Arithmetic Problems," *Journal of Educational Research*, 38. 86–96, October, 1944.
14. Swenson, E. J., "Difficulty Ratings of Addition Facts as Related to Learning Method," *Journal of Educational Research*, 38. 81–85, October, 1944.
15. Tuck, G. I., "Drill Problems on Functions," *School* (Secondary Edition), 33. 345–346, December, 1944.
16. Wilson, G. M., "Basic Considerations for Profitable Research in Arithmetic," *Journal of Educational Research*, 38. 119–123, October, 1944.

NEWS NOTES

Joseph Clifton Brown, since 1929 superintendent of schools, Pelham, (N. Y.), died, following an operation, January 16, at the age of sixty-five years. Dr. Brown was head of the department of mathematics (1910–15), Horace Mann School, Teachers College, Columbia University; assistant professor of education (1915–16), University of Illinois; president (1916–27), State Teachers College (St. Cloud, Minn.); and president (1927–29), Northern Illinois State Teachers College, De Kalb. He was the author, with the late Dr. Coffman, of "How to Teach Arithmetic," and, singly or in collaboration, of several textbooks in elementary mathematics. He recently published two small booklets, "Easy Arithmetical Short Cuts," and "Easy Tricks With Numbers," which had a very wide sale.

W. H. Hill retired November 1, 1944 as Professor of Mathematics in the Kansas State Teachers College of Pittsburg, Kansas. He came to the institution in 1921 as an assistant professor. He became an associate professor in 1934, and a full professor in 1936. Since 1925 he has had complete charge of the training of teachers of secondary mathematics. During these years, 283 seniors have received valuable assistance in preparation for the teaching of secondary mathematics. His devoted service to the teaching profession is greatly appreciated by his colleagues and students with whom he has been associated.

Professor Hill is a member of the National Council of Teachers of Mathematics and the Kansas Association of Teachers of Mathematics. He was a charter member of the Mathematical Association of America.

The fifth meeting of the Men's Mathematics Club of Chicago and the Metropolitan Area was held on Feb. 16. Dr. J. M. Kinney of Wilson Junior College spoke on the topic "A Brief History of the Function Concept in Secondary Mathematics," and W. G. Hendershot, of Roosevelt High School on "A Program for Teaching the Function Concept in High School Algebra."

Roster of Presidents of the Men's Mathematics Club of Chicago and Metropolitan Area from Its Organization in 1914 to 1945

Compiled by Edwin W. Schreiber

1. Charles M. Austin, 1914–16, Oak Park High School, Oak Park, Ill.
2. John R. Clark, 1916–18, Chicago Normal College, Chicago, Ill.
3. William W. Gorsline, 1918–20, Crane Technical High School, Chicago, Ill.
4. *Marquis J. Newell (d. '41), 1920–21, Evanston Township High School, Evanston, Ill.
5. *Horace C. Wright (d. ?), 1921–22, University High School, Chicago, Ill.
6. Everett W. Owen, 1922–23, Oak Park High School, Oak Park, Ill.
7. Olice Winter, 1923–24, Harrison Technical High School, Chicago, Ill.
8. Edwin W. Schreiber, 1924–25, Proviso Township High School, Maywood, Ill.
9. Marx E. Holt, 1925–26, Seward School, Chicago, Ill.
10. Orion M. Miller, 1926–27, Chicago Normal College, Chicago, Ill.
11. Edgar S. Leach, 1927–28, Evanston Township High School, Evanston, Ill.
12. John T. Johnson, 1928–29, Chicago Normal College, Chicago, Ill.
13. *William H. Clark (d. '41), 1929–30, Lindbloom High School, Chicago, Ill.
14. Charles Leckrone, 1930–31, Lake View High School, Chicago, Ill.
15. Edgar C. Hinkle, 1931–32, Board of Examiners, Chicago, Ill.
16. Walter S. Pope, 1932–33, Morton Township High School, Cicero, Ill.
17. Francis W. Runge, 1933–34, Evanston Township High School, Evanston, Ill.
18. Joseph J. Urbancek, 1934–35, Lane Technical High School, Chicago, Ill.
19. William C. Krathwohl, 1935–36, Armour Institute of Technology, Chicago, Ill.
20. J. Russel McDonald, 1936–37, Morton Township High School, Cicero, Ill.
21. *D. Talmage Petty (d. '42), 1937–38, Francis W. Parker School, Chicago, Ill.
22. Joel S. Georges, 1938–39, Wright Junior College, Chicago, Ill.
23. George E. Hawkins, 1939–40, University High School, Chicago, Ill.
24. Hans Gutekunst, 1940–41, Batavia High School, Batavia, Ill.
25. Charles E. Jenkins, 1941–42, Foreman High School, Chicago, Ill.
26. Samuel F. Bibb, 1942–43, Illinois Institute of Technology, Chicago, Ill.
27. Milton D. Oestreicher, 1943–44, Francis W. Parker School, Chicago, Ill.
28. Glenn F. Hewitt, 1944–45, Von Steuben High School, Chicago, Ill.

The institution given is at time of presidency.
* Deceased.

The following letter has recently been received at the office of The Mathematics Teacher:

311 W. McKay St.
Carlsbad, N. M.
Jan. 1, 1945

Editorial Office
The Mathematics Teacher
525 W. 120th St.
New York 27, N. Y.

Gentlemen:

I am enclosing my check for $2 for membership in the *National Council of Teachers of Mathematics.* I am doing this because I have found your publication, The Mathematics Teacher, which is being sent to the Carlsbad High School where I am now teaching, so interesting and helpful that I should like to have copies of my own for reference. You are to be congratulated on the uniformly high standard of the articles secured for this publication.

Sincerely yours,
Max Werminghaus

We hope that we shall receive many more such letters.—Editor.

NEW BOOKS

Allen, E. B., Dis Maly, and S. H. Starkey, Jr., *Vital Mathematics*. The Macmillan Co., New York. 1944. 456 pp.

Andres, Paul G., and Haim Reingold, *Basic Mathematics for Engineers*. John Wiley and Sons, Inc., Chapman and Hall, Ltd., London. 1944. 670 pp. $4.00.

Ballou and Steen, *Plane and Spherical Trigonometry*. Ginn and Company, 1943. 172 pp. $2.20.

Ballou and Steen, *Spherical Trigonometry with Tables*. Ginn and Company, 1943. 173+84 pp. $1.25.

Blackburn, E. F., *Basic Air Navigation*. McGraw-Hill Book Company, Inc., New York, 1944. 300 pp. $3.00.

Bradley and Upton, *Air Navigation Workbook*. American Book Co., 1943. 112 pp. $0.88.

Brown, Kenneth E., *General Mathematics in American Colleges*. Bureau of Publications, Teachers College, Columbia Univ., New York, 1944. 156 pp. $2.35.

Davis, Dale S., *Chemical Engineering Nomographs*. McGraw-Hill Book Company, New York, 1944. 211 pp. $2.50.

Daus, Gleason, and Whyburn, *Basic Mathematics for War and Industry*. The Macmillan Company, N. Y., 1944. 271 pp. $2.00.

Doole, Howard P., *Plane and Spherical Trigonometry*. Thomas Y. Crowell Company, New York, 1944. 183 pp. $1.75.

Encyclopedia of Modern Education, edited by Rivlin and Schueler, H. Hubner and Co., Inc., 1943. 902 pp.

Freilich, Aaron, Henry H. Shanholt and Joseph Seidlin, *Spherical Trigonometry*. Silver Burdett Co., New York, 1944. 118 pp. $1.28.

Gray, Hugh and Lillian R. Lieber, *The Education of T. C. Mits*. Norton and Co., New York, 1944. 230 pp. $2.50.

Greitzer, Samuel L., *Elementary Topography and Map Reading*. McGraw-Hill Book Company, Inc., 1944. 150 pp. $1.60.

Hadel, Walter, *Celestial Navigation*. McGraw-Hill Book Company, New York, 1944. 261 pp. $2.50.

Hadel, Walter, *Celestial Navigation: A Problem Manual*. McGraw-Hill Book Company, New York, 1944. 256 pp. $2.50.

Hanson, Paul P., *Military Applications of Mathematics*. McGraw-Hill Book Company, 1944. 445 pp. $2.40.

Hart, William L., *Plane and Spherical Trigonometry*. D. C. Heath & Company, Boston, 1943. 124 pp.

Jordan, Pascual, *Physics of the 20th Century*. The Philosophical Library of New York, 1944. 185 pp. $4.00.

Kells, Kern, Bland, and Orleans, *Elements of Trigonometry*. McGraw-Hill Book Company, Inc., 1943. 363 pp.

Klaf, A. A., *Calculus Refresher*. McGraw-Hill Book Company, New York, 1944. 421 pp. $3.00.

Naidich, James F., *Air Navigation Made Easy*. McGraw-Hill Book Company, New York, 1944. 124 pp. $1.75.

Northrop, Eugene P., *Riddles in Mathematics*. D. Van Nostrand Company, New York, 1944. 226 pp. plus appendix. $3.00.

Nyberg, Joseph A., *Fundamentals of Plane Geometry*. American Book Company, New York, 1944. 359 pp. $1.48.

Nyberg, Joseph A., *Fundamentals of Algebra*. American Book Company, New York, 1944. 336 pp. $1.40.

Rivlin, Harry N., Editor, and Schueler, Herbert, Assoc. Editor, *Encyclopedia of Modern Education*, The Philosophical Library of New York, 1943.

Stewart, John Q., and Newton L. Pierce, *Marine and Air Navigation*. Ginn and Company, Boston, 1944. 465 pp. $4.50.

The Philosophy of Bertrand Russell, Editor, Paul Arthur Schlipp. Northwestern U., 1944. (The Library of Living Philosophies, Vol. V.) 815 pp. $4.00.

Upton, Clifford B., *Short Course in Computation*. American Book Company, New York, 1944. 176 pp.

Waugh, Albert E., *Elements of Statistical Method*. McGraw-Hill Book Company, New York, 1943.